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Quantum-limited position detection and amplification: A linear response perspective

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Using standard linear response relations, we derive the quantum limit on the sensitivity of a *generic* linear-response position detector, and the noise temperature of a generic linear amplifier. Particular emphasis is placed on the detector's effective temperature and damping effects; the former quantity directly determines the dimensionless power gain of the detector. Unlike the approach used in the seminal work of Caves [Phys. Rev. D **26** 1817 (1982)], the linear-response approach directly involves the noise properties of the detector, and allows one to derive simple necessary and sufficient conditions for reaching the quantum limit. Our results have direct relevance to recent experiments on nanoelectromechanical systems, and complement recent theoretical studies of particular mesoscopic position detectors.

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I. INTRODUCTION

Recent advances in the field of nanoelectromechanical (NEMS) systems have renewed interest in the question of quantum limited detection and amplification;[1–8] several recent experiments have even come close to achieving this ideal limit.[5,7] For position measurements, the quantum limit corresponds to the maximum sensitivity allowed by quantum mechanics of a weak, continuous measurement.[9,10] For amplifiers, the quantum limit refers to the minimum amount of noise that must be added by a high-gain linear amplifier to the input signal.[11,12] Despite the fact that quantum constraints on amplifiers have been studied and understood for quite some time, there still seems to be some confusion in the NEMS and mesoscopics communities as to the precise definition of the quantum limit, and on its origin. It has even been conjectured that it may be possible to beat the quantum limit using a weakly coupled, continuously measuring mesoscopic detector.[2] Much of the confusion here stems from the fact that the seminal work on quantum limited amplification by Caves[11] uses a description that is difficult to apply directly to the mesoscopic detectors presently in use (i.e., a single-electron transistor or quantum point contact).

In this paper, we approach the question of quantum limited detection and amplification using nothing more than standard linear-response (Kubo) relations. This approach has the advantage that it *directly* involves the noise properties of the detector, and allows one to derive simple conditions that a detector must satisfy in order to reach the quantum limit. Achieving the quantum limit is seen to require a detector with "ideal" noise properties, a requirement that many detectors (e.g., a SET in the sequential tunneling regime[13–15]) fail to meet. We demonstrate that the quantum limit for displacement detection and amplification is analogous to the quantum limit constraining quantum nondemolition (QND) measurements of a qubit.[13–20] This is despite the fact that in the present problem, the detector-system coupling is not QND–the coupling Hamiltonian does not commute with the Hamiltonian of the input system, and thus back-action force noise results in additional output noise at later times. We place a special emphasis on the effective temperature and damping effects of the detector; in amplifier language, the latter corresponds to the amplifier's input and output impedances. We find that the detector's effective temperature directly determines the dimensionless power gain of the detector, and also constrains correlations between the detector's intrinsic output noise and back-action force noise. The approach presented here sheds light on recent findings which show that the effects of an out-of-equilibrium detector on an oscillator can be described via an effective damping coefficient and temperature,[1,4,6] we show that this is a generic feature of weakly coupled linear response detectors.

Finally, turning to the specific case of position detection of an oscillator, we find that to reach the quantum limit on the displacement sensitivity with a large power gain, the damping of the oscillator must be predominantly independent of the detector. We also show that optimizing the displacement sensitivity (the quantity measured in the experiments of Refs. 5 and 7) is *not* the same as minimizing the smallest detectable force.[6] Note that linear-response constraints on position measurements were also considered briefly by Averin in Ref. 16, though that work did not consider the role of detector-dependent damping or the detector's effective temperature, two crucial elements of the work presented here.

II. BASICS

For definiteness, we start by considering the case of a generic linear-response detector measuring the position of a harmonic oscillator; the almost equivalent case of a generic linear amplifier will be discussed in Sec. V. Our generic position detector has an input and output port, each of which is characterized by an operator ($\hat{F}$ and $\hat{I}$, respectively). The input operator $\hat{F}$ is linearly coupled to the position $\hat{x}$ of the oscillator

$$H_{\text{int}} = -A\hat{F}\cdot\hat{x}, \tag{1}$$

where A is the dimensionless coupling strength and $A\cdot\hat{F}$ is nothing more than the back-action force associated with the measurement. The expectation value of the output operator $\hat{I}$

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(e.g., current) responds to the motion of the oscillator; we assume throughout that the coupling is weak enough that one can use linear response, and thus we have

$$\Delta\langle\hat{I}(t)\rangle = A\int_{-\infty}^{\infty} dt'\,\lambda(t-t')\langle\hat{x}(t')\rangle, \tag{2}$$

where λ is the detector gain given by the Kubo formula

$$\lambda(t-t') = -\frac{i}{\hbar}\,\theta(t-t')\langle[\hat{I}(t),\hat{F}(t')]\rangle. \tag{3}$$

The expectation value here is over the (stationary) zero-coupling density matrix ρ_0 of the detector. Neither this state nor the Hamiltonian of the detector need to be specified in what follows.

A. Effective environment for oscillator

Turning to the oscillator, we assume that it is coupled both to the detector and to an equilibrium Ohmic bath with temperature T_{bath}. The bath models intrinsic (i.e., detector-independent) dissipation of the oscillator. For a weak coupling to the detector $(A\to 0)$, one can calculate the oscillator Keldysh Green functions using lowest-order-in-A perturbation theory. One finds that at this level of approximation, the full quantum dynamics of the oscillator is described by a classical-looking Langevin equation (see Appendix A and Ref. 20):

$$m\ddot{x}(t) = -m\Omega^2 x(t) - \gamma_0\dot{x}(t) - A^2\int dt'\,\gamma(t-t')\dot{x}(t') + F_0(t) + A\cdot F(t). \tag{4}$$

In this Langevin equation, $x(t)$ is a classically fluctuating quantity, not an operator. Its average value, as determined from Eq. (4), corresponds to the full quantum-mechanical expectation of the operator $\hat{x}(t)$. Similarly, the noise in $x(t)$ calculated from Eq. (4) corresponds precisely to $\bar{S}_x(t) = \langle\{\hat{x}(t),\hat{x}(0)\}\rangle/2$, the symmetrized noise in the quantum operator $\hat{x}$ (see Appendix A for the details of this correspondence). Here, Ω is the renormalized frequency of the oscillator, m its renormalized mass, γ_0 describes damping due to the equilibrium bath, and $F_0(t)$ is the corresponding fluctuating force. The spectrum of the F_0 fluctuations are given by the standard equilibrium relation

$$\bar{S}_{F_0}(\omega) = \gamma_0\hbar\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{5}$$

The remaining terms in Eq. (4) describe the influence of the detector—$A\cdot F(t)$ is the random back-action force produced by the detector, while $A^2\gamma$ describes damping due to the detector. The spectrum of the $F(t)$ fluctuations is given by the symmetrized force noise of the detector

$$\bar{S}_F(\omega) \equiv \frac{1}{2}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle\{\hat{F}(t),\hat{F}(0)\}\rangle \tag{6}$$

while for $\gamma(t)$, one has

$$\gamma(\omega) = \frac{-\operatorname{Im}\lambda_F(\omega)}{\omega} \equiv \frac{\operatorname{Re}\int_0^{\infty} dt\langle[\hat{F}(t),\hat{F}(0)]\rangle e^{i\omega t}}{\hbar\omega} = \frac{1}{\hbar}\frac{S_F(\omega)-S_F(-\omega)}{2\omega}, \tag{7}$$

where λ_F is the linear response susceptibility describing the response of F to a change in x, and $S_F(\omega)$ is the (unsymmetrized) detector F-noise, calculated at zero coupling:

$$S_F(\omega) = \int_{-\infty}^{\infty} dt\langle\hat{F}(t)\hat{F}(0)\rangle e^{i\omega t}. \tag{8}$$

Thus, though the detector is not assumed to be in equilibrium, by treating its coupling to the oscillator to lowest order, we obtain a simple Langevin equation in which the detector provides both damping and a fluctuating force. Note that in general, the detector force noise $\bar{S}_F(\omega)$ will not be related to $\gamma(\omega)$ via the temperature, as would hold for an equilibrium system [i.e., Eq. (5)]. However, for a given frequency ω we can define the effective temperature $T_{\mathrm{eff}}(\omega)$ via

$$\coth\left(\frac{\hbar\omega}{2k_BT_{\mathrm{eff}}(\omega)}\right) \equiv \frac{S_F(\omega)+S_F(-\omega)}{S_F(\omega)-S_F(-\omega)}. \tag{9}$$

In the $\omega\to 0$ limit this reduces to

$$2k_BT_{\mathrm{eff}} = \left.\frac{\bar{S}_F}{\gamma}\right|_{\omega=0}. \tag{10}$$

Note that T_{eff} is by no means equal to the physical temperature of the detector, nor does it correspond to the "noise temperature" of the detector (see Sec. V); the effective temperature only serves as a measure of the asymmetry of the detector's quantum noise. For example, it has been found for SET and tunnel junction detectors that $k_BT_{\mathrm{eff}}\simeq \mathrm{eV}$, where V is the source-drain voltage of the detector.[3,4,6]

B. Detector output noise

Next, we link fluctuations in the position of the oscillator to noise in the output of the detector. On a strictly classical level, we would treat both the oscillator position $x(t)$ and the detector output $I(t)$ as classically fluctuating quantities. Using the linearity of the detector's response, we could then write

$$\delta I_{\mathrm{total}}(\omega) = \delta I_0(\omega) + A\lambda(\omega)\cdot\delta x(\omega). \tag{11}$$

The first term (δI_0) describes the intrinsic (oscillator-independent) fluctuations in the detector output, while the second term corresponds to the amplified fluctuations of the oscillator. These are in turn given by Eq. (4):

$$\delta x(\omega) = -\left[\frac{1/m}{(\omega^2-\Omega^2)+i\omega\Omega/Q(\omega)}\right][F_0(\omega)+A\cdot F(\omega)]$$
$$\equiv -g(\omega)[F_0(\omega)+A\cdot F(\omega)], \tag{12}$$

where $Q(\omega)=m\Omega/[\gamma_0+\gamma(\omega)]$ is the oscillator quality factor. It follows that the total noise in the detector output is given *classically* by

$$S_{I,\text{tot}}(\omega) = S_I(\omega) + |g(\omega)|^2|\lambda(\omega)|^2[A^4 S_F(\omega) + A^2 S_{F_0}(\omega)] - 2A^2\,\text{Re}[g(\omega)S_{IF}(\omega)]. \tag{13}$$

Here, S_I, S_F, and S_{IF} are the (classical) detector noise correlators calculated in the absence of any coupling to the oscillator.

To apply the classically derived Eq. (13) to our quantum detector-plus-oscillator system, we interpret $S_{I,\text{tot}}$ as the total symmetrized quantum-mechanical output noise of the detector, and simply substitute in the right-hand side the symmetrized quantum-mechanical detector noise correlators $\bar{S}_F$, $\bar{S}_I$, and $\bar{S}_{IF}$, defined as in Eq. (6). Though this may seem rather ad hoc, one can easily demonstrate that Eq. (13) thus interpreted would be completely rigorous, quantum mechanically, *if* the detector correlation functions obeyed Wick's theorem. Thus, quantum corrections to Eq. (13) will arise from the non-Gaussian nature of the detector noise correlators. We expect from the central limit theorem that such corrections will be small in the relevant limit where ω is much smaller that the typical detector frequency $\sim k_B T_{\text{eff}}/\hbar$, and neglect these corrections in what follows. Note that the validity of Eq. (13) for a specific model of a tunnel junction position detector has been verified in Ref. 21.

C. Quantum constraint on detector noise

We turn now to the fundamental constraint on the detector noise correlators and gain λ which, in the present approach, serves as the basis of the quantum limit. We have

$$\bar{S}_I(\omega)\bar{S}_F(\omega) \geq \frac{\hbar^2}{4}\{\text{Re}[\lambda(\omega)-\lambda'(\omega)]\}^2 + [\text{Re}\,\bar{S}_{IF}(\omega)]^2. \tag{14}$$

Here, λ' is the reverse gain of the detector (i.e., the gain in an experiment where we couple x to I and attempt to measure F). Equation (14) tells us that if our detector has gain and no positive feedback $(\text{Re}\lambda\cdot\text{Re}\lambda'\leq 0)$, then it must have a minimum amount of back-action and output noise. This equation was proved rigorously in Ref. 18 for $\omega=0$ using a Schwartz inequality; the proof given there may be straightforwardly generalized to finite ω if one now uses symmetrized-in-frequency noise correlators. Note that almost all mesoscopic detectors that have been studied in detail (i.e., a SET or generalized quantum point contact) have been found to have $\lambda'=0$.[18]

We now define a quantum-limited detector at frequency ω as having a minimal amount of noise at ω, that is, it satisfies

$$\bar{S}_I(\omega)\bar{S}_F(\omega) = \frac{\hbar^2}{4}\{\text{Re}[\lambda(\omega)]\}^2 + [\text{Re}\,\bar{S}_{IF}(\omega)]^2. \tag{15}$$

This is the same condition that arose in the study of quantum-limited qubit detection,[18,19] with the exception that in that case, one also required that $\text{Re}\,\bar{S}_{IF}=0$. We will show that Eq. (15) must indeed be satisfied in order to achieve the quantum limit on position sensitivity, or the quantum limit on the noise temperature of an amplifier.

As discussed in Ref. 18, having a quantum-limited detector [i.e., satisfying Eq. (15)] implies a tight connection between the input and output ports of the detector. Note first that we may write each symmetrized noise correlators as a sum over transitions $|i\rangle\rightarrow|f\rangle$, e.g.,

$$\bar{S}_F(\omega) = \pi\hbar\sum_{i,f}\langle i|\rho_0|i\rangle|\langle f|F|i\rangle|^2[\delta(E_f-E_i+\omega) + \delta(E_f-E_i-\omega)], \tag{16}$$

where ρ_0 is the stationary detector density matrix, and $|i/f\rangle$ is a detector eigenstate with energy $E_{i/f}$. To achieve the "ideal noise" condition of Eq. (15) at frequency ω, there must exist a complex factor α (having dimensions $[I]/[F]$) such that

$$\langle f|I|i\rangle = \alpha\langle f|F|i\rangle \tag{17}$$

for *each* pair of initial and final states $|i\rangle$, $|f\rangle$ contributing to $\bar{S}_F(\omega)$ and $\bar{S}_I(\omega)$ [cf. Eq. (16)]. Note that this *not* the same as requiring Eq. (17) to hold for all possible states $|i\rangle$ and $|f\rangle$. In Ref. 18, the requirement of Eq. (17) was further interpreted as implying that there is no additional information regarding the input signal that is available in the detector but not revealed in its output $\hat{I}$.

For a quantum limited detector with $\lambda'=0$, the coefficient α in Eq. (17) can be found from

$$|\alpha(\omega)|^2 = \bar{S}_I(\omega)/\bar{S}_F(\omega) \tag{18}$$

$$\tan[\arg\,\alpha(\omega)] = -\frac{\hbar\,\text{Re}\,\lambda(\omega)/2}{\text{Re}\,\bar{S}_{IF}(\omega)}. \tag{19}$$

Thus, a nonvanishing gain implies that $\text{Im}\,\alpha\neq 0$. It then follows from Eq. (17) and the hermiticity of $\hat{I}$, $\hat{F}$ that for a quantum limited detector, the set of all initial states $|i\rangle$ contributing to the noise has no overlap with the set of all final states $|f\rangle$. This immediately implies that a quantum-limited detector cannot be in equilibrium.

D. Power gain

To be able to say that our detector truly amplifies the motion of the oscillator, the power delivered by the detector to a following amplifier must be much larger than the power used to drive the oscillator—i.e., the detector must have a dimensionless power gain $G_P(\omega)$ much larger than one. As we now show, this requirement places further constraints on the effective temperature and noise properties of the detector.

In what follows, we consider the simple (and usual) case where there is no reverse gain $\lambda'=0$. The power gain $G_P(\omega)$

of our generic position detector may be defined as follows. Imagine first that we drive the oscillator with a force $F_D \cos \omega t$; this will cause the output of our detector $\langle I(t)\rangle$ to also oscillate at frequency ω. To optimally detect this signal in the detector output, we further couple I to a second oscillator with natural frequency ω and position y: $H'_{\rm int}=B\hat{I}\cdot\hat{y}$. The oscillations in I will now act as a driving force on the auxiliary oscillator y. $G_P(\omega)$ is then defined as the *maximum* ratio between the power provided to the output oscillator y from the detector, versus the power fed into the input of the amplifier. This ratio is maximized if there is no intrinsic (i.e., detector independent) damping of either the input or output oscillators. The damping of of the input oscillator is then completely given by $A^2\gamma$ [see Eq. (7)], whereas the damping of the output oscillator $B^2\gamma_{\rm out}$ is given by

$$\gamma_{\rm out}(\omega) = -\frac{\operatorname{Im}\lambda_I(\omega)}{\omega} \equiv \frac{\operatorname{Re}\int_0^\infty dt \langle[\hat{I}(t),\hat{I}(0)]\rangle e^{i\omega t}}{\hbar\omega}. \quad (20)$$

Using an overbar to denote a time average, we then have

$$G_P(\omega) \equiv \frac{\overline{P_{\rm out}}}{\overline{P_{\rm in}}} = \frac{B^2\gamma_{\rm out}(\omega)\cdot\overline{\dot{y}^2}}{A^2\gamma(\omega)\cdot\overline{\dot{x}^2}} \quad (21)$$

$$= \frac{B^2\gamma_{\rm out}(\omega)}{A^2\gamma(\omega)} \frac{\left[\left(\frac{\omega}{\omega B^2\gamma_{\rm out}(\omega)}\right)[B\cdot A|\lambda(\omega)|\cdot|g(\omega)|F_{\rm ext}]\right]^2}{[\omega|g(\omega)|F_{\rm ext}]^2}$$
$$= \frac{|\lambda(\omega)|^2}{\omega^2\gamma_{\rm out}(\omega)\cdot\gamma(\omega)} = \frac{|\lambda(\omega)|^2}{\operatorname{Im}\lambda_F(\omega)\cdot\operatorname{Im}\lambda_I(\omega)}. \quad (22)$$

Thus, the power gain is a simple dimensionless ratio formed by the three different response coefficients characterizing the detector. It is completely analogous to the power gain of a voltage amplifier [see Eq. (48) of Sec. V].

Turning now to the important case of a quantum limited detector, that is a detector satisfying the ideal noise condition of Eq. (15), we find that the expression for the power gain can be further simplified using Eq. (17). One finds

$$G_P(\omega) = \frac{(\operatorname{Im}\alpha)^2\coth\left(\frac{\hbar\omega}{2k_BT_{\rm eff}}\right) + (\operatorname{Re}\alpha)^2}{|\alpha|^2/4}. \quad (23)$$

It thus follows that to have $G_P \gg 1$, one needs $\mathbf{k}_B T_{\rm eff} \gg \hbar\omega$: a large power gain implies a large effective detector temperature. In the large G_P limit, we have

$$G_P \simeq \left[\frac{\operatorname{Im}\alpha}{|\alpha|}\frac{4k_BT_{\rm eff}}{\hbar\omega}\right]^2. \quad (24)$$

Finally, an additional consequence of the large $G_P(\omega)$, large $T_{\rm eff}$ limit is that the imaginary parts of the gain $\lambda(\omega)$ and cross correlator $\overline{S_{IF}}(\omega)$ become negligible; they are suppressed by the small factor $\hbar\omega/k_BT_{\rm eff}$. This is shown explicitly in Appendix B.

III. QUANTUM LIMIT ON ADDED DISPLACEMENT NOISE

The sensitivity of a position detector is determined by the added displacement noise $S_x(\omega)$, which is simply the total detector contribution to the noise in the detector's output, referred back to the oscillator. It is this quantity which has been probed in recent experiments,[5,7] and which has a fundamental quantum constraint, as we derive below.

To define $S_x(\omega)$, we first introduce $S_{x,\rm tot}(\omega)$, which is simply the total noise in the output of the detector [$S_{I,\rm tot}(\omega)$, cf. Eq. (13)] referred back to the oscillator

$$S_{x,\rm tot}(\omega) \equiv \frac{S_{I,\rm tot}(\omega)}{A^2\lambda^2}. \quad (25)$$

We then separate out detector-dependent contributions to $S_{x,\rm tot}(\omega)$ from the intrinsic equilibrium fluctuations of the oscillator; the added displacement noise $S_x(\omega)$ is defined as the former quantity

$$S_{x,\rm tot}(\omega) \equiv S_x(\omega) + \frac{\gamma_0}{\gamma_0 + A^2\gamma}S_{x,\rm eq}(\omega,T), \quad (26)$$

where

$$S_x(\omega) = \frac{\overline{S_I}}{|\lambda|^2A^2} + A^2|g(\omega)|^2\overline{S_F} - \frac{2\operatorname{Re}[\lambda^* g^*(\omega)\overline{S_{IF}}]}{|\lambda|^2}, \quad (27)$$

$$S_{x,\rm eq}(\omega,T) = \hbar\coth\left(\frac{\hbar\omega}{2k_BT}\right)[-\operatorname{Im} g(\omega)] \quad (28)$$

and where we have omitted writing the ω dependence of the noise correlators and λ. Here, $S_{x,\rm eq}(\omega,T)$ represents the equilibrium fluctuations that would result if *all* the damping were due to the equilibrium bath; its contribution to $S_{x,\rm tot}(\omega)$ is reduced, as only part of the damping results from the bath.

We now proceed to derive the quantum limit on $S_x(\omega)$. Examining Eq. (27) for $S_x(\omega)$, and ignoring for a moment the detector-dependent damping of the oscillator, we see that the first term (i.e., the intrinsic detector output noise referred back to the detector input) is proportional to $1/A^2$, while the second term (i.e., the back-action of the detector) scales as A^2. We would thus expect $S_x(\omega)$ to attain a minimum value at an optimal choice of coupling $A=A_{\rm opt}$, where both these terms make equal contributions. Using the inequality $X^2+Y^2 \geq 2XY$ we see that this value serves as a lower bound on S_x even in the presence of detector-dependent damping. Defining $\phi=\arg g(\omega)$, we thus have the bound

$$S_x(\omega) \geq 2|g(\omega)|\left[\sqrt{\overline{S_I}\,\overline{S_F}/|\lambda|^2} - \frac{\operatorname{Re}[\lambda^* e^{-i\phi(\omega)}\overline{S_{IF}}]}{|\lambda|^2}\right] \quad (29)$$

where the minimum value is achieved when

$$A^2_{\rm opt} = \sqrt{\frac{\overline{S_I}(\omega)}{|\lambda(\omega)g(\omega)|^2\overline{S_F}(\omega)}}. \quad (30)$$

In the case where the detector-dependent damping is negligible, the RHS of this equation is independent of A, and thus Eq. (30) can be satisfied by simply tuning the coupling strength A; in the more general case where there is detector-dependent damping, the RHS is also a function of A, and it may no longer be possible to achieve Eq. (30) by simply tuning A.

While Eq. (29) is certainly a bound on the added displacement noise $S_x(\omega)$, it does not in itself represent the quantum limit. Reaching the quantum limit requires more than simply balancing the detector back-action and intrinsic output noises [i.e., the first two terms in Eq. (27)]; one also needs a detector with ideal noise properties, that is a detector which satisfies Eq. (15) and thus the proportionality condition of Eq. (17). Using the quantum noise constraint of Eq. (14) to further bound $S_x(\omega)$, we obtain

$$S_x(\omega) \geqslant 2\frac{|g(\omega)|}{|\lambda|}\left[\sqrt{\left(\frac{\hbar\,\mathrm{Re}\,\lambda}{2}\right)^2 + (\mathrm{Re}\,\bar{S}_{IF})^2} - \frac{\mathrm{Re}[\lambda^* e^{-i\phi(\omega)}\bar{S}_{IF}]}{|\lambda|^2}\right]. \tag{31}$$

The minimum value of $S_x(\omega)$ in Eq. (31) is now achieved when one has *both* an optimal coupling [i.e., Eq. (30)] and a quantum limited detector, that is one which satisfies Eq. (15). Note again that an arbitrary detector will not satisfy the ideal noise condition of Eq. (15).

Next, we consider the relevant case where our detector is a good amplifier and has a power gain $G_P(\omega) \gg 1$ over the width of the oscillator resonance. As discussed in Appendix B, this implies that we may neglect the imaginary parts of λ and $\bar{S}_{IF}$, as they are suppressed by $\hbar\Omega/k_B T_{\mathrm{eff}} \ll 1$. We then have

$$S_x(\omega) \geqslant 2|g(\omega)|\left[\sqrt{\left(\frac{\hbar}{2}\right)^2 + \left(\frac{\bar{S}_{IF}}{\lambda}\right)^2} - \frac{\cos[\phi(\omega)]\bar{S}_{IF}}{\lambda}\right]. \tag{32}$$

Finally, as there is no further constraint on $\bar{S}_{IF}/\lambda$, we can minimize the expression over its value. The minimum $S_x(\omega)$ is achieved for a detector whose cross-correlator satisfies

$$\left.\frac{\bar{S}_{IF}(\omega)}{\lambda}\right|_{\mathrm{optimal}} = \frac{\hbar}{2}\cot\phi(\omega), \tag{33}$$

with the minimum value being given by

$$S_x(\omega)|_{\min} = \hbar|\mathrm{Im}\,g(\omega)| = \lim_{T\to 0} S_{x,\mathrm{eq}}(\omega,T), \tag{34}$$

where $S_{x,\mathrm{eq}}(\omega,T)$ is the equilibrium contribution to $S_{x,\mathrm{tot}}(\omega)$ defined in Eq. (28). Thus, in the limit of a large power gain, we have that at each frequency, the minimum displacement noise due to the detector is precisely equal to the noise arising from a zero temperature bath. This conclusion is irrespective of the strength of the intrinsic (detector-independent) oscillator damping.

The result of Eq. (34) is essentially identical to the conclusion of Caves,[11] who found that a high-gain amplifier (modelled as a set of bosonic modes and a scattering matrix) must add at least $\hbar\omega/2$ of noise to an input signal at frequency ω. Here, our input signal corresponds to the damped oscillator, and the minimum value of $S_x(\omega)$ in Eq. (34) corresponds precisely to the zero-point noise of the damped oscillator.

Though it reaches a similar conclusion, the linear-response approach has several advantages over the approach of Caves. First, we do not have to model our detector as a set of bosonic modes and a scattering matrix, something that is difficult to do for many mesoscopic detectors. More significantly, the linear-response approach makes explicitly clear what is needed to reach the quantum limit. We find that to reach the quantum limit on the added displacement noise $S_x(\omega)$, one needs

(i) A quantum limited detector, that is a detector which satisfies the "ideal noise" condition of Eq. (15), and hence the proportionality condition of Eq. (17).

(ii) A coupling A which satisfies Eq. (30).

(iii) A detector cross-correlator $\bar{S}_{IF}$ which satisfies Eq. (33).

Note that condition (i) is identical to what is required for quantum-limited detection of a qubit; it is rather demanding, and requires that there is no "wasted" information about the input signal in the detector which is not revealed in the output.[18] Also note that $\cot\phi$ changes quickly as a function of frequency across the oscillator resonance, whereas $\bar{S}_{IF}$ will be roughly constant; condition (ii) thus implies that it will not be possible to achieve a minimal $S_x(\omega)$ across the entire oscillator resonance. A more reasonable goal is to optimize S_x at resonance, $\omega = \Omega$. As $g(\Omega)$ is imaginary, Eq. (33) tells us that $\bar{S}_{IF}$ should be zero. Assuming we have a quantum-limited detector with a large power gain ($k_B T_{\mathrm{eff}} \gg \hbar\Omega$), the remaining condition on the coupling A [Eq. (30)] may be written as

$$\frac{A_{\mathrm{opt}}^2\gamma}{\gamma_0 + A_{\mathrm{opt}}^2\gamma} = \left|\frac{\mathrm{Im}\,\alpha}{\alpha}\right|\frac{1}{\sqrt{G_P(\Omega)}} = \frac{\hbar\Omega}{4k_B T_{\mathrm{eff}}}. \tag{35}$$

As $A^2\gamma$ is the detector-dependent damping of the oscillator, we thus have that to achieve the quantum-limited value of $S_x(\Omega)$ with a large power gain, one needs the intrinsic damping of the oscillator to be much larger than the detector-dependent damping. The detector-dependent damping must be small enough to compensate the large effective temperature of the detector; if the bath temperature satisfies $\hbar\Omega/k_B \ll T_{\mathrm{bath}} \ll T_{\mathrm{eff}}$, Eq. (35) implies that at the quantum limit, the temperature of the oscillator will be given by

$$T_{\mathrm{osc}} \equiv \frac{A^2\gamma\cdot T_{\mathrm{eff}} + \gamma_0\cdot T_{\mathrm{bath}}}{A^2\gamma + \gamma_0} \to \frac{\hbar\Omega}{4k_B} + T_{\mathrm{bath}}. \tag{36}$$

Thus, at the quantum limit and for large T_{eff}, the detector raises the oscillator's temperature by $\hbar\Omega/4k_B$. As expected, this additional heating is only *half* the zero point energy; in contrast, the quantum-limited value of $S_x(\omega)$ corresponds to the full zero point result, as it also includes the contribution of the intrinsic output noise of the detector.

Finally, we remark that if one did not assume $k_B T_{\mathrm{eff}} \gg \hbar\Omega$ as is needed for a large power gain, we would have to keep the imaginary parts of λ and $\bar{S}_{IF}$. One can show that for $k_B T_{\mathrm{eff}}/\hbar\Omega \to 0$, it is possible to have a perfect anticorrelation between the intrinsic detector output noise δI_0 and the back-action force δF, and thus have $S_x(\omega)=0$. Thus, similar to the results of Caves,[11] in the limit of unit power gain (i.e., small detector effective temperature), there is no quantum limit on S_x, as perfect anticorrelations between the two kinds of detector noise (i.e., in $\hat{I}$ and $\hat{F}$) are possible.

IV. QUANTUM LIMIT ON FORCE SENSITIVITY

In this section, we now ask a different question: what is the smallest magnitude force acting on the oscillator that can be detected with our displacement detector?[6] This force sensitivity is also subject to a quantum limit; as with the quantum limit on $S_x(\omega)$, one again needs a quantum-limited detector [i.e., one satisfying the ideal noise condition of Eq. (15)] to reach the maximal sensitivity. However, the conditions on the coupling A (i.e., the detector-dependent damping) are quite different than what is needed to optimize $S_x(\omega=\Omega)$; in particular, one can reach the quantum limit on the force sensitivity even if there is no intrinsic oscillator damping.

We start by imagining that a force $F_{\text{ext}}(t)=\Delta p\cdot\delta(t)$ acts on our oscillator, producing a change $\Delta I(t)$ in the output of our detector. The corresponding signal-to-noise ratio is defined as[6,9]

$$S/N=\int\frac{d\omega}{2\pi}\frac{|\Delta I(\omega)|^2}{S_{I,\text{tot}}(\omega)}=(\Delta p)^2\int\frac{d\omega}{2\pi}\frac{|\lambda(\omega)g(\omega)|^2}{S_{I,\text{tot}}(\omega)}. \quad (37)$$

In what follows, we make the reasonable assumption that the detector noise correlators and gain are frequency independent over the width of the oscillator resonance. We also take the relevant limit of a large power gain (i.e., $k_BT_{\text{eff}}\gg\hbar\Omega$), and assume a best-case scenario where $T_{\text{bath}}=0$ and where the total oscillator quality factor $Q\gg1$. Using Eq. (13), one finds that the maximal S/N is indeed obtained for a quantum-limited detector [i.e., one satisfying Eq. (15)]:

$$S/N\leq\frac{2(\Delta p)^2}{\hbar m\tilde{\Omega}}\int_0^\infty\frac{dx}{\pi}\frac{\Gamma}{(x^2-1)^2+\Gamma^2(1+\Lambda^2x)}, \quad (38)$$

where equality occurs for a quantum-limited detector. For such a detector, we have

$$\tilde{\Omega}^2=\Omega^2-\left(\frac{1}{Q_{\text{det}}}\right)\left(\frac{4k_BT_{\text{eff}}}{\hbar\Omega}\right)\left(\frac{\text{Im}\,\alpha\,\text{Re}\,\alpha}{|\alpha|^2}\right), \quad (39)$$

$$\Gamma=\left(\frac{1}{Q_{\text{det}}}\right)\left(\frac{4k_BT_{\text{eff}}}{\hbar\Omega}\right)\left(\frac{\text{Im}\,\alpha}{|\alpha|}\frac{\Omega}{\tilde{\Omega}}\right)^2, \quad (40)$$

$$\Lambda^2=\left(\frac{\gamma_0}{A^2\gamma}\right)\left(\frac{\hbar\tilde{\Omega}}{4k_BT_{\text{eff}}}\right)\left(\frac{|\alpha|}{\text{Im}\,\alpha}\right), \quad (41)$$

where $Q_{\text{det}}=m\Omega/(A^2\gamma)$ is the quality factor corresponding to the detector-dependent damping. It is clear from Eq. (38) that the signal to noise ratio can be further maximized by having both $\Gamma\ll1$ and $\Lambda\ll1$. This requires

$$\frac{\gamma_0}{A^2\gamma}\ll\frac{k_BT_{\text{eff}}}{\hbar\Omega}\ll Q_{\text{det}}. \quad (42)$$

If this condition is satisfied, it follows that $\tilde{\Omega}\simeq\Omega$, and we have

$$S/N\leq\frac{(\Delta p)^2}{\hbar M\Omega}. \quad (43)$$

Demanding now $S/N\geq1$, we find that the minimum detectable Δp is $\sqrt{\hbar M\Omega}$, $\sqrt{2}$ times the zero point value. Note that the requirement of Eq. (42) on the coupling A is very different from what is needed to reach the quantum limit on $S_x(\Omega)$ [see Eq. (35)]. In the present case, it is possible to reach the quantum limit on the force sensitivity even if the damping from the detector dominates ($A^2\gamma\gg\gamma_0$); in contrast, it is impossible to achieve the quantum limit on $S_x(\Omega)$ in this regime.

V. QUANTUM LIMIT ON THE NOISE TEMPERATURE OF A VOLTAGE AMPLIFIER

In this final section, we generalize the discussion of the previous sections to the case of a generic linear voltage amplifier (see, e.g., Ref. 12); the quantum limit on the noise temperature of the amplifier is seen to be analogous to the quantum limit on the added displacement noise $S_x(\omega)$.

As with the position detector, the voltage amplifier is characterized by an input operator $\hat{Q}$ and an output operator $\hat{V}$; these play the role, respectively, of $\hat{F}$ and $\hat{I}$ in the position detector. $\hat{V}$ represents the output voltage of the amplifier, while $\hat{Q}$ is the operator which couples to the input signal $v_{\text{in}}(t)$ via a coupling Hamiltonian

$$H_{\text{int}}=v_{\text{in}}(t)\cdot\hat{Q}. \quad (44)$$

In more familiar terms, $\tilde{I}_{\text{in}}=-d\hat{Q}/dt$ represents the current flowing into the amplifier. We also assume that the output of the amplifier ($\hat{V}$) is connected to an external circuit via a term

$$H'_{\text{int}}=q_{\text{out}}(t)\cdot\hat{V}, \quad (45)$$

where $\tilde{i}_{\text{out}}=dq_{\text{out}}/dt$ is the current in the external circuit. In what follows, we will use quantities defined in the previous sections for a position detector, simply substituting in $\hat{I}\to\hat{V}$ and $\hat{F}\to\hat{Q}$.

Similar to the position detector, there are three response coefficients for the amplifier: the voltage gain coefficient λ [see Eq. (3)], the Q-Q susceptibility λ_Q which determines damping at the input [see Eq. (7)], and the V-V susceptibility λ_V which determines damping at the output [see Eq. (20)]. The diagonal susceptibilities determine the input and output impedances:

$$Z_{\text{in}}(\omega)=[i\omega\lambda_Q(\omega)]^{-1}, \quad (46)$$

$$Z_{\mathrm{out}}(\omega) = \frac{\lambda_V(\omega)}{-i\omega}, \tag{47}$$

i.e., $\langle \tilde{I}_{\mathrm{in}} \rangle_\omega = [1/Z_{\mathrm{in}}(\omega)] v_{\mathrm{in}}(\omega)$ and $\langle V \rangle_\omega = Z_{\mathrm{out}}(\omega) \tilde{i}_{\mathrm{out}}(\omega)$, where the subscript ω indicates the Fourier transform of a time-dependent expectation value.

We will consider throughout this section the case of no reverse gain, $\lambda' = 0$. We may then use Eq. (22) for the power gain; it has the expected form

$$G_P = \lambda^2 \frac{\mathrm{Re}\, Z_{\mathrm{in}}}{\mathrm{Re}\, Z_{\mathrm{out}}} = \frac{\langle V \rangle^2 / \mathrm{Re}\, Z_{\mathrm{out}}}{(v_{\mathrm{in}})^2 / \mathrm{Re}\, Z_{\mathrm{in}}}. \tag{48}$$

Finally, we may again define the effective temperature $T_{\mathrm{eff}}(\omega)$ of the amplifier via Eq. (9), and define a quantum-limited amplifier as one which satisfies the ideal noise condition of Eq. (15). For such an amplifier, the power gain will again be determined by the effective temperature via Eq. (23).

Turning to the noise, we introduce $S_{v,\mathrm{tot}}(\omega)$, the total noise at the output port of the amplifier referred back to the input. Assuming the voltage source producing the input signal v_{in} has an impedance Z_S and a temperature $T_S \gg \hbar\omega / k_B$, we may write

$$S_{v,\mathrm{tot}}(\omega) = 2 k_B T_s \,\mathrm{Re}\, Z_S(\omega) + S_v(\omega). \tag{49}$$

The first term is the equilibrium noise associated with the signal source, while the second term is the total amplifier contribution to the noise at the output port, referred back to the input. Taking the limit of a large power gain (which ensures λ, $\overline{S}_{VQ}$ are real), we have [see Eq. (27) and Ref. 12]:

$$S_v(\omega) = \frac{\overline{S}_V}{\lambda^2} + |\tilde{Z}|^2 [\omega^2 \overline{S}_Q] + 2 \,\mathrm{Im}\, \tilde{Z} \frac{[w \overline{S}_{VQ}]}{\lambda}, \tag{50}$$

$$\tilde{Z} = \frac{Z_S Z_{\mathrm{in}}}{Z_S + Z_{\mathrm{in}}}, \tag{51}$$

where for clarity, we have dropped the ω dependence of the noise correlators, gain, and impedances. The first term in Eq. (50) represents the intrinsic output noise of the amplifier, while the second term represents the back-action of the amplifier: fluctuations in the input current $\hat{I}_{\mathrm{in}} = -d\hat{Q}/dt$ lead to fluctuations in the voltage drop across the parallel combination of the source impedance Z_s and the input impedance Z_{in}, and hence in the signal going into the amplifier. The last term in Eq. (50) represents correlations between these two sources of noise. We see that the general form of Eq. (50) is completely analogous to that for the added displacement noise $S_x(\omega)$ of a displacement detector, see Eq. (27).

We are now ready to introduce the noise temperature of our amplifier: T_N is defined as the amount we must increase the temperature T_S of the voltage source to account for the noise added by the amplifier,[22], i.e., we wish to rewrite Eq. (49) as

$$S_{v,\mathrm{tot}}(\omega) \equiv 2 k_B (T_S + T_N) \mathrm{Re}\, Z_s(\omega). \tag{52}$$

In what follows, we assume that $|Z_s| \ll |Z_{\mathrm{in}}|$, which means that Z_{in} drops out of Eq. (50); we will test the validity of this assumption at the end. Using Eq. (50), and writing $Z_s = |Z_s| e^{i\phi}$, we have immediately

$$2 k_B T_N = \frac{1}{\cos\phi} \left[\frac{\overline{S}_V}{|Z_s| \lambda^2} + |Z_s| (\omega^2 \overline{S}_Q) \right] + 2 \tan\phi \frac{\omega \overline{S}_{VQ}}{\lambda}. \tag{53}$$

To derive a bound on T_N, we first perform the classical step of optimizing over the the magnitude and phase of the source impedance $Z_S(\omega)$; this is in contrast to the optimization of $S_x(\omega)$, where one would optimize over the strength of the coupling. We find

$$k_B T_N \geqslant \omega \sqrt{\frac{\overline{S}_V(\omega) \overline{S}_Q(\omega) - [\overline{S}_{VQ}(\omega)]^2}{[\lambda(\omega)]^2}}, \tag{54}$$

where the minimum is achieved for an optimal source impedance satisfying

$$|Z_s(\omega)|_{\mathrm{opt}} = \sqrt{\frac{\overline{S}_V(\omega)/[\lambda(\omega)]^2}{\omega^2 \overline{S}_Q(\omega)}}, \tag{55}$$

$$\sin\phi(\omega)|_{\mathrm{opt}} = -\frac{\overline{S}_{VQ}(\omega)}{\sqrt{\overline{S}_V(\omega) \overline{S}_Q(\omega)}}. \tag{56}$$

As with the displacement sensitivity, simply performing a classical optimization (here, over the choice of source impedance) is not enough to reach the quantum limit on T_N. One also needs to have an amplifier which satisfies the ideal noise condition of Eq. (15). Using this equation, we obtain the final bound

$$k_B T_N \geqslant \frac{\hbar\omega}{2}, \tag{57}$$

where the minimum corresponds to both having optimized Z_s and having an amplifier whose noise satisfies Eq. (15).

Finally, we need to test our initial assumption that $|Z_S| \ll |Z_{\mathrm{in}}|$. Using the proportionality condition of Eq. (17) and the fact that we are in the large power gain limit $[G_P(\omega) \gg 1]$, we find

$$\left| \frac{Z_S(\omega)}{\mathrm{Re}\, Z_{\mathrm{in}}(\omega)} \right| = \left| \frac{\alpha}{\mathrm{Im}\, \alpha} \right| \frac{\hbar\omega}{4 k_B T_{\mathrm{eff}}} = \frac{1}{\sqrt{G_P(\omega)}} \ll 1. \tag{58}$$

It follows that $|Z_S| \ll |Z_{\mathrm{in}}|$ in the large power gain, large effective temperature regime of interest, and our neglect of $|Z_{\mathrm{in}}|$ in Eq. (53) is justified. Eq. (58) is analogous to the case of the displacement detector, where we found that reaching the quantum limit on resonance required the detector-dependent damping to be much weaker than the intrinsic damping of the oscillator [see Eq. (35)].

Thus, similar to the situation of the displacement detector, the linear response approach allows us both to derive rigorously the quantum limit on the noise temperature T_N of an amplifier, and to state conditions that must be met to reach this limit. To reach the quantum-limited value of T_N with a large power gain, one needs *both* a tuned source impedance Z_S, *and* an amplifier which possesses ideal noise properties [see Eqs. (15) and (17)].

VI. CONCLUSIONS

In this paper we have derived the quantum limit on position measurement of an oscillator by a generic linear response detector, and on the noise temperature of a generic linear amplifier. The approach used makes clear what must be done to reach the quantum limit; in particular, one needs a detector or amplifier satisfying the ideal noise constraint of Eq. (15), a demanding condition which is not satisfied by most detectors. Our treatment has emphasized both the damping effects of the detector and its effective temperature T_{eff}; in particular, we have found that the requirement of a large detector power gain translates into a requirement of a large detector effective temperature.

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APPENDIX A: DERIVATION OF LANGEVIN EQUATION

In this Appendix, we prove that an oscillator weakly coupled to an arbitrary out-of-equilibrium detector is described by the Langevin equation given in Eq. (4), an equation which associates an effective temperature and damping kernel to the detector; a similar perturbative approach for the problem of a qubit coupled to a detector was considered by Shnirman *et al.* in Ref. 20

We start by defining the oscillator matrix Keldysh Green function

$$\check{G}(t) = \begin{pmatrix} G^K(t) & G^R(t) \\ G^A(t) & 0 \end{pmatrix}, \tag{A1}$$

where $G^R(t-t') = -i\theta(t-t')\langle[x(t),x(t')]\rangle$, $G^A(t-t') = i\theta(t'-t) \times\langle[x(t),x(t')]\rangle$, and $G^K(t-t') = -i\langle\{x(t),x(t')\}\rangle$. At zero coupling to the detector ($A=0$), the oscillator is only coupled to the equilibrium bath, and thus $\check{G}_0$ has the standard equilibrium form

$$\check{G}_0(\omega) = \frac{\hbar}{m}\begin{pmatrix} -2\,\mathrm{Im}\, g_0(\omega)\coth\left(\frac{\hbar\omega}{2k_B T_{\mathrm{bath}}}\right) & g_0(\omega) \\ g_0(\omega)^* & 0 \end{pmatrix}, \tag{A2}$$

where

$$g_0(\omega) = \frac{1}{\omega^2 - \Omega^2 + i\omega\gamma_0/m} \tag{A3}$$

and where γ_0 is the intrinsic damping coefficient, and T_{bath} is the bath temperature.

We next treat the effects of the coupling to the detector in perturbation theory. Letting $\check{\Sigma}$ denote the corresponding self-energy, the Dyson equation for $\check{G}$ has the form

$$[\check{G}(\omega)]^{-1} = [\check{G}_0(\omega)]^{-1} - \begin{pmatrix} 0 & \Sigma^A(\omega) \\ \Sigma^R(\omega) & \Sigma^K(\omega) \end{pmatrix}. \tag{A4}$$

To lowest order in A, $\check{\Sigma}(\omega)$ is given by

$$\check{\Sigma}(\omega) = A^2\check{D}(\omega) \tag{A5}$$

$$\equiv \frac{A^2}{\hbar}\int dt e^{i\omega t}\begin{pmatrix} 0 & i\theta(-t)\langle[\hat{F}(t),\hat{F}(0)]\rangle \\ -i\theta(t)\langle[\hat{F}(t),\hat{F}(0)]\rangle & -i\langle\{\hat{F}(t),\hat{F}(0)\}\rangle \end{pmatrix}. \tag{A6}$$

Using this lowest-order self energy, Eq. (A4) yields

$$G^R(\omega) = \frac{\hbar}{m(\omega^2-\Omega^2) - A^2\,\mathrm{Re}\,D^R(\omega) + i\omega[\gamma_0 + \gamma(\omega)]}, \tag{A7}$$

$$G^A(\omega) = [G^R(\omega)]^*, \tag{A8}$$

$$G^K(\omega) = -2i\,\mathrm{Im}\,G^R(\omega) \times \frac{\gamma_0\coth\left(\frac{\hbar\omega}{2k_B T_{\mathrm{bath}}}\right) + \gamma(\omega)\coth\left(\frac{\hbar\omega}{2k_B T_{\mathrm{eff}}}\right)}{\gamma_0 + \gamma(\omega)}, \tag{A9}$$

where $\gamma(\omega)$ is given by Eq. (7), and $T_{\mathrm{eff}}(\omega)$ is defined by Eq. (9). The main effect of the real part of the retarded F Green function $D^R(\omega)$ in Eq. (A7) is to renormalize the oscillator frequency Ω and mass m; we simply incorporate these shifts into the definition of Ω and m in what follows.

If $T_{\mathrm{eff}}(\omega)$ is frequency independent, then Eqs. (A7)–(A9) for $\hat{G}$ corresponds exactly to an oscillator coupled to two equilibrium baths with damping kernels γ_0 and $\gamma(\omega)$. The correspondence to the Langevin equation (4) is then immediate. In the more general case where $T_{\mathrm{eff}}(\omega)$ has a frequency dependence, the correlators $G^R(\omega)$ and $G^K(\omega)$ are in exact correspondence to what is found from the Langevin equation Eq. (4): $G^K(\omega)$ corresponds to symmetrized noise calculated from Eq. (4), while $G^R(\omega)$ corresponds to the response coefficient of the oscillator calculated from Eq. (4). This again proves the validity of using the Langevin equation (4) to calculate the oscillator noise in the presence of the detector to lowest order in A.

APPENDIX B: SUPPRESSION OF IMAGINARY PARTS OF λ AND $\overline{S_{IQ}}$

Defining $S_{IF}(\omega) = \int_{-\infty}^{\infty} dt\langle I(t)F(0)\rangle$, one has the relations

$$\hbar[\lambda(\omega) - \lambda'(\omega)^*] = -i[S_{IF}(\omega) - S_{IF}(-\omega)], \tag{B1}$$

$$\overline{S_{IF}}(\omega) = \tfrac{1}{2}[S_{IF}(\omega) + S_{IF}(-\omega)^*] \tag{B2}$$

which follow directly from the definitions of $\overline{S_{IF}}$ and λ. Assuming now that we have a quantum limited detector [i.e., Eq. (15) is satisfied], a vanishing reverse gain ($\lambda'=0$), and $k_B T_{\rm eff} \gg \hbar\omega$, we can use Eqs. (9) and (17) in conjunction with the above equations to show

$$\hbar\lambda(\omega) = -\gamma(\omega)[4k_B T_{\rm eff} \mathrm{Im}\, \alpha + i2\hbar\Omega\, \mathrm{Re}\, \alpha], \tag{B3}$$

$$\overline{S_{IF}}(\omega) = \gamma(\omega)[2k_B T_{\rm eff}\, \mathrm{Re}\, \alpha - i\hbar\Omega\, \mathrm{Im}\, \alpha]. \tag{B4}$$

We see immediately that the imaginary parts of λ and $\overline{S_{IF}}$ are suppressed compared to the corresponding real parts by a small factor $\hbar\Omega/2k_B T_{\rm eff}$.

[1] M. P. Blencowe and M. N. Wybourne, Appl. Phys. Lett. **77**, 3845 (2000).
[2] Y. Zhang and M. P. Blencowe, J. Appl. Phys. **91**, 4249 (2002).
[3] A. D. Armour, M. P. Blencowe, and Y. Zhang, Phys. Rev. B **69**, 125313 (2004).
[4] D. Mozyrsky and I. Martin, Phys. Rev. Lett. **89**, 018301 (2002).
[5] R. G. Knobel and A. N. Cleland, Nature (London) **424**, 291 (2003).
[6] D. Mozyrsky, I. Martin, and M. B. Hastings, Phys. Rev. Lett. **92**, 018303 (2004).
[7] M. D. LaHaye, O. Buu, B. Camarota, and K. C. Schwab, Science **304**, 74 (2004).
[8] N. M. Chtchelkatchev, W. Belzig, and C. Bruder, cond-mat/0401486 (unpublished).
[9] V. B. Braginsky, and F. Y. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
[10] C. M. Caves, K. S. Thorne, W. P. Drever, V. D. Sandberg, and N. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[11] C. M. Caves, Phys. Rev. D **26**, 1817 (1982).
[12] M. H. Devoret and R. J. Schoelkopf, Nature (London) **406**, 1039 (2000).
[13] A. Shnirman and G. Schön, Phys. Rev. B **57**, 15 400 (1998).
[14] A. N. Korotkov, Phys. Rev. B **63**, 115403 (2001).
[15] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).
[16] D. V. Averin, cond-mat/0004364 (unpublished).
[17] S. Pilgram and M. Büttiker, Phys. Rev. Lett. **89**, 200401 (2002).
[18] A. A. Clerk, S. M. Girvin, and A. D. Stone, Phys. Rev. B **67**, 165324 (2003).
[19] D. V. Averin, cond-mat/0301524 (unpublished).
[20] A. Shnirman, D. Mozyrksy, and I. Martin, cond-mat/0311325 (unpublished).
[21] A. A. Clerk and S. M. Girvin, Phys. Rev. B **70**, 121303(R) (2004).
[22] Note that our definition of the noise temperature T_N conforms with that of Devoret and Schoelkopf (Ref. 12), but is slightly different than that of Caves (Ref. 11). Caves assumes the source is initially at zero temperature, and consequently uses the full quantum expression for its equilibrium noise. In contrast, we have assumed that $k_B T_S \gg \hbar\omega$. The different definition used by Caves leads to the result $k_B T_N \geq \hbar\omega/(\ln 3)$ as opposed to our Eq. (57).