Quantum noise and quantum measurement

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Introduction

The fact that quantum mechanics can place restrictions on our ability to make measurements is something we all encounter in our first quantum mechanics class. One is typically presented with the example of the Heisenberg microscope (Heisenberg, 1930), where the position of a particle is measured by scattering light off it. The smaller the wavelength of light used, the better the precision of the measurement. However, decreasing the wavelength also increases the magnitude of backaction disturbance of the particle's momentum by the measurement (i.e. the momentum kick delivered to the particle by the scattering event). One finds that the imprecision of the measurement Δx_{imp} and the backaction momentum disturbance Δp_{BA} of the particle are constrained in a way suggestive of the usual Heisenberg uncertainty principle

$$\Delta x_{\mathrm{imp}} \cdot \Delta p_{\mathrm{BA}} \gtrsim h. \tag{1.1}$$

As instructive as this example is, it hardly provides a systematic way for formulating precise quantum limits on measurement in more general settings. Recent progress in the general area of engineered quantum systems has rekindled interest in such fundamental limits on measurement and amplification. One would like to have a precise formulation of what these constraints are, and moreover, an understanding of how one can achieve these ultimate limits in realistic and experimentally-relevant setups. In these lectures, I will describe an extremely useful and powerful method for doing this, the so-called "quantum noise" approach. The focus will be on weak, continuous measurements, where the measured system is only weakly coupled to the detector, and information is obtained only gradually in time. Even on a purely classical level, such measurements are limited by the presence of detector noise. Quantum mechanically, one finds that there are fundamental quantum limits on the noise properties of any system capable of acting as a detector or amplifier. These constraints then directly yield quantum limits on various measurement tasks.

For the most part, these notes draw extensively from material presented in our recent review article (Clerk *et al.*, 2010), but do not attempt to be as exhaustive as that work. I have slightly reworked the discussion of several key points to hopefully add clarity. I also include in these notes a few topics not found in the review article. These include a discussion of the quantum shot noise of a quantum point contact QPC (Sec. 2.3), a formulation of the quantum limit on QND qubit detection beyond weak-coupling (Sec. 3.5), and a heuristic discussion of how a QPC can miss that limit due to correlated backaction-imprecision noise (Sec. 3.3).

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Quantum noise spectral densities: some essential features

In this chapter, we give a compact (and no doubt highly incomplete) review of some basic properties of spectral densities describing quantum noise.

2.1 Classical noise basics

Consider a classical random signal $I(t)$. The signal is characterized by zero mean $\langle I(t)\rangle = 0$, and autocorrelation function

$$G_{II}(t,t') = \langle I(t)I(t')\rangle. \tag{2.1}$$

The autocorrelation function is analogous to a covariance matrix: for $t = t'$, it tells us the variance of the fluctuations of $I(t)$, where as for $t \neq t'$, it tells us if and how fluctuations of $I(t)$ are correlated with those at $I(t')$. Some crucial concepts regarding noise are:

- *Stationary noise*. The statistical properties of the fluctuations are time-translation invariant, and hence $G_{II}(t,t') = G_{II}(t-t')$.
- *Gaussian fluctuations*. The noise is fully characterized by its autocorrelation function; there are no higher-order cumulants.
- *Correlation time*. This time-scale τ_{c} governs the decay of $G_{II}(t)$: $I(t)$ and $I(t')$ are uncorrelated (i.e. $G_{II}(t-t') \to 0$) when $|t-t'| \gg \tau_c$.

For stationary noise, it is often most useful to think about the fluctuations in the frequency domain. In the same way that $I(t)$ is a Gaussian random variable with zero mean, so is its Fourier transform, which we define as:

$$I_T[\omega] = \frac{1}{\sqrt{T}} \int_{-T/2}^{+T/2} dt\, e^{i\omega t} I(t), \tag{2.2}$$

where T is the sampling time. In the limit $T \gg \tau_{\mathrm{c}}$ the integral is a sum of a large number $N \approx \frac{T}{\tau_{\mathrm{c}}}$ of random uncorrelated terms. We can think of the value of the integral as the end point of a random walk in the complex plane which starts at the origin. Because the distance traveled will scale with $\sqrt{T}$, our choice of normalization makes the statistical properties of $I[\omega]$ independent of the sampling time T (for sufficiently large T). Notice that $I_T[\omega]$ has the peculiar units of $[I]\sqrt{\mathrm{secs}}$ which is usually denoted $[I]/\sqrt{\mathrm{Hz}}$.

The spectral density of the noise (or power spectrum) $\mathcal{S}_{II}[\omega]$ answers the question "how big is the noise at frequency ω?". It is simply the variance of $I_T(\omega)$ in the large-time limit:

$$\mathcal{S}_{II}[\omega] \equiv \lim_{T\to\infty} \langle |I_T[\omega]|^2 \rangle = \lim_{T\to\infty} \langle I_T[\omega] I_T[-\omega] \rangle. \tag{2.3}$$

A reasonably straightforward manipulation (known as the Wiener-Khinchin theorem) tells us that the spectral density is equal to the Fourier transform of the autocorrelation function

$$\mathcal{S}_{II}[\omega] = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} G_{II}(t). \tag{2.4}$$

We stress that Eq. (2.3) provides a simple intuitive understanding of what a spectral density represents, whereas in theoretical calculations, one almost always starts with the expression in Eq. (2.4). We also stress that since the autocorrelation function $G_{II}(t)$ is real, $\mathcal{S}_{II}[\omega] = \mathcal{S}_{II}[-\omega]$. This is of course in keeping with Eq. (2.2), which tells us that negative and positive frequency components of the noise are related by complex conjugation, and hence necessarily have the same magnitude.

As a simple example, consider a simple harmonic oscillator of mass M and frequency Ω. The oscillator is maintained in equilibrium with a large heat bath at temperature T via some infinitesimal coupling which we will ignore in considering the dynamics. The solution of Hamilton's equations of motion gives

$$x(t) = x(0)\cos(\Omega t) + p(0)\frac{1}{M\Omega}\sin(\Omega t), \tag{2.5}$$

where $x(0)$ and $p(0)$ are the (random) values of the position and momentum at time $t = 0$. It follows that the position autocorrelation function is

$$\begin{aligned} G_{xx}(t) &= \langle x(t)x(0) \rangle \\ &= \langle x(0)x(0) \rangle \cos(\Omega t) + \langle p(0)x(0) \rangle \frac{1}{M\Omega}\sin(\Omega t). \end{aligned} \tag{2.6}$$

Classically in equilibrium there are no correlations between position and momentum. Hence the second term vanishes. Using the equipartition theorem $\frac{1}{2}M\Omega^2\langle x^2 \rangle = \frac{1}{2}k_{\mathrm{B}}T$, we arrive at

$$G_{xx}(t) = \frac{k_{\mathrm{B}}T}{M\Omega^2}\cos(\Omega t), \tag{2.7}$$

which leads to the spectral density

$$\mathcal{S}_{xx}[\omega] = \pi\frac{k_{\mathrm{B}}T}{M\Omega^2}\left[\delta(\omega - \Omega) + \delta(\omega + \Omega)\right], \tag{2.8}$$

which is indeed symmetric in frequency.

2.2 Quantum noise spectral densities

2.2.1 Definition

In formulating quantum noise, one turns from a noisy classical signal $I(t)$ to a Heisenberg-picture Hermitian operator $\hat{I}(t)$. Similar to our noisy classical signal, one needs to think about measurements of $\hat{I}(t)$ statistically. One can thus introduce a quantum-noise spectral density which completely mimics the classical definition, e.g.:

$$S_{xx}[\omega] = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \langle \hat{x}(t)\hat{x}(0)\rangle. \tag{2.9}$$

We have simply inserted the quantum autocorrelation function in the classical definition. The expectation value is the quantum statistical average with respect to the noisy system's density matrix; we assume that this is time-independent, which then also gives us an autocorrelation function which is time-translational invariant.

What makes quantum noise so quantum? There are at least three answers to this question that we will explore in turn:

- *Zero-point motion.* While a classical system at zero-temperature has no noise (c.f. Eq. (2.8)), quantum mechanically there are still fluctuations, i.e. $S_{xx}[\omega]$ need not be zero.
- *Frequency asymmetry.* Quantum mechanically, $\hat{x}(t)$ and $\hat{x}(t')$ need not commute when $t \neq t'$. As a result the autocorrelation function $\langle \hat{x}(t)\hat{x}(t')\rangle$ can be complex, and $S_{xx}[\omega]$ need not equal $S_{xx}[-\omega]$. This of course can never happen for a classical noise spectral density.
- *Heisenberg constraints.* For any system that can act as a detector or amplifier, there are fundamental quantum constraints that bound its noise. These constraints have their origin in the uncertainty principle, and have no classical counterpart.

Let's start our discussion with the second point above, and make things concrete by again considering the example of a harmonic oscillator in thermal equilibrium. We again assume that the oscillator is maintained in equilibrium with a large heat bath at temperature T via some infinitesimal coupling which we will ignore in considering the dynamics. The solutions of the Heisenberg equations of motion are the same as for the classical case but with the initial position x and momentum p replaced by the corresponding quantum operators. It follows that the position autocorrelation function is

$$\begin{aligned} G_{xx}(t) &= \langle \hat{x}(t)\hat{x}(0)\rangle \\ &= \langle \hat{x}(0)\hat{x}(0)\rangle \cos(\Omega t) + \langle \hat{p}(0)\hat{x}(0)\rangle \frac{1}{M\Omega} \sin(\Omega t). \end{aligned} \tag{2.10}$$

Unlike the classical case, the second term on the RHS cannot be zero: that would violate the commutation relation $[\hat{x}(0), \hat{p}(0)] = i\hbar$. Writing $\hat{x}$ and $\hat{p}$ in terms of ladder operators, one finds that $\langle \hat{x}(0)\hat{p}(0)\rangle = i\hbar/2$: it is purely imaginary. One shouldn't be too troubled by this, as $\hat{x}(0)\hat{p}(0)$ is not Hermitian and hence is not an observable quantity. Evaluating $\langle \hat{x}(0)^2\rangle$ in a similar manner, we find

$$G_{xx}(t) = x_{\mathrm{ZPF}}^2 \left\{ n_{\mathrm{B}}(\hbar\Omega) e^{+i\Omega t} + [n_{\mathrm{B}}(\hbar\Omega) + 1] e^{-i\Omega t} \right\}, \tag{2.11}$$

where $x_{\mathrm{ZPF}}^2 \equiv \hbar/2M\Omega$ is the RMS zero-point uncertainty of x in the quantum ground state, and n_{B} is the Bose-Einstein occupation factor. Fourier transforming then yields a spectral density that is asymmetric in frequency:

$$S_{xx}[\omega] = 2\pi x_{\mathrm{ZPF}}^2 \left\{ n_{\mathrm{B}}(\hbar\Omega)\delta(\omega+\Omega) + [n_{\mathrm{B}}(\hbar\Omega)+1]\delta(\omega-\Omega) \right\}. \tag{2.12}$$

Note that in the high temperature limit $k_{\mathrm{B}}T \gg \hbar\Omega$ we have $n_{\mathrm{B}}(\hbar\Omega) \sim n_{\mathrm{B}}(\hbar\Omega) + 1 \sim \frac{k_{\mathrm{B}}T}{\hbar\Omega}$. Thus, in this limit the $S_{xx}[\omega]$ becomes symmetric in frequency as expected classically, and coincides with the classical expression for the position spectral density (cf. Eq. (2.8)).

The Bose-Einstein factors suggest a way to understand the frequency-asymmetry of Eq. (2.12): the positive frequency part of the spectral density has to do with stimulated emission of energy *into* the oscillator and the negative frequency part of the spectral density has to do with emission of energy *by* the oscillator. That is, the positive frequency part of the spectral density is a measure of the ability of the oscillator to *absorb* energy, while the negative frequency part is a measure of the ability of the oscillator to *emit* energy.

The above interpretation is of course not restricted to harmonic oscillators or thermal states. Consider now the quantum noise associated with a general observable $\hat{I}(t)$, and let $\hat{\rho}$ be the system's density matrix. We will also let $|j\rangle$ denote the system's energy eigenstates (eigenenergy E_j), where for simplicity we take j to be a discrete index. Letting $\hat{U}(t)$ denote the time evolution operator, the quantum noise spectral density $S_{II}[\omega]$ can be written

$$\begin{aligned} S_{II}[\omega] \equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle \hat{I}(t)\hat{I}(0)\rangle &= \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\mathrm{Tr}\,\left[\hat{\rho}\,\hat{U}^{\dagger}(t)\hat{I}(0)\hat{U}(t)\hat{I}(0)\right] \\ &= 2\pi\hbar\sum_{i,j}\langle i|\hat{\rho}|i\rangle \left|\langle j|\hat{I}|i\rangle\right|^2 \delta(\hbar\omega - (E_j - E_i)). \end{aligned} \tag{2.13}$$

We have used the fact that since $\hat{\rho}$ is time-independent, it must be diagonal in the basis of energy eigenstates. The above expression is a standard Lehman representation for a quantum correlation function. It is nothing more than a sum of Fermi Golden rule transition rates (from the state $|i\rangle$ to $|j\rangle$), mediated by the operator $\hat{I}$. More explicitly, if we coupled $\hat{I}$ to a qubit such that

$$H_{\mathrm{qb}} = \frac{\hbar\Omega}{2}\hat{\sigma}_z + A\hat{I}\hat{\sigma}_x, \tag{2.14}$$

and then treated the second term via Fermi's Golden rule, the qubit excitation rate Γ_+ and relaxation rate Γ_- would be simply given by:

$$\Gamma_{\pm} = \frac{A^2}{\hbar^2}S_{II}[\mp\Omega]. \tag{2.15}$$

The origin of the frequency asymmetry is exactly the same as in our oscillator example. For $\omega > 0$, the spectral density describes transitions where the noise source absorbs energy, whereas for $\omega < 0$, it describes transitions where energy is emitted by the noise source. The noise at positive and negative frequencies need not be equal, as in general, absorption will occur at a higher rate than emission. In the extreme limit of zero temperature, where the system is in its ground state, we see that there is still noise at positive frequencies, as the noise source can still absorb energy.

Finally, we note that in thermal equilibrium $\langle i|\hat{\rho}|i\rangle \propto \exp(-E_i/k_B T)$, which necessarily relates the positive and negative frequency noise via

$$S_{II}[\omega]/S_{II}[-\omega] = \exp[\hbar\omega/k_B T], \tag{2.16}$$

where T is the temperature of the noise source. On a physical level, this ensures that the transition rates in Eq. (2.15) obey detailed balance, and thus cause the qubit to also relax to an equilibrium state at the same temperature as the noise source

2.2.2 Classical interpretation

Our analysis so far seems to indicate that apart from the formal similarity in their definitions (c.f. Eqs. (2.4) and Eq (2.9)), classical and quantum noise spectral densities have very little in common. The classical noise spectral density tells us the 'size' of the noise at a particular frequency, whereas in contrast, the quantum noise spectral density tells us the magnitude of Golden rule transition rates for emission or absorption events. The frequency-asymmetry of a quantum noise spectral density also gives the appearance that it contains more information than one has classically. One is tempted to conclude that referring to $S_{II}[\omega]$ as noise is an abuse of terminology.

To overcome these apprehensions and to gain a deeper understanding of quantum noise spectral densities, it is useful to think of our noisy quantity as a force $\hat{F}$, and see what happens when we weakly couple this force to a harmonic oscillator, i.e.:

$$\hat{H}_{\mathrm{int}} = -\hat{x}\hat{F}. \tag{2.17}$$

Classically, including this force in Newton's equation yields a Langevin equation:

$$M\ddot{x} = -M\Omega^2 x - M\gamma_{\mathrm{cl}}\dot{x} + F_{\mathrm{cl}}(t). \tag{2.18}$$

In addition to the noisy force, we have included a damping term (rate γ_{cl}). This will prevent the oscillator from being infinitely heated by the noise source; we can think of it as describing the *average* value of the force exerted on the oscillator by the noise source, which is now playing the role of a dissipative bath. If this bath is in thermal equilibrium at temperature T, we also expect the oscillator to equilibrate to the same temperature. This implies that the heating effect of $F_{\mathrm{cl}}(t)$ must be precisely balanced by the energy-loss effect of the damping force. More explicitly, one can use Eq. (2.18) to derive an equation for the average energy of the oscillator $\langle E\rangle$. As we are assuming a weak coupling between the bath and the oscillator, we can take $\gamma_{\mathrm{cl}} \ll \Omega$, and hence find

$$\frac{d}{dt}\langle E\rangle = -\gamma_{\mathrm{cl}}\langle E\rangle + \frac{\mathcal{S}_{FF}[\Omega]}{2M}. \tag{2.19}$$

Insisting that the stationary $\langle E\rangle$ obey equipartition then leads directly to the classical fluctuation dissipation relation:

$$\mathcal{S}_{FF}[\omega] = 2M\gamma_{\mathrm{cl}}k_B T. \tag{2.20}$$

Let's now look at our problem quantum mechanically. Writing $\hat{x}$ in terms of ladder operators, we see that $\hat{H}_{\mathrm{int}}$ will cause transitions between different oscillator Fock states. Treating $\hat{H}_{\mathrm{int}}$ in perturbation theory, we thus derive Fermi Golden rule transition rates $\Gamma_{n\pm1,n}$ for transitions from the n to the $n \pm 1$ Fock state:

$$\Gamma_{n+1,n} = (n+1)\frac{x_{\mathrm{ZPF}}^2}{\hbar^2} S_{FF}[-\Omega] \equiv (n+1)\Gamma_{\uparrow}, \tag{2.21}$$

$$\Gamma_{n-1,n} = (n)\frac{x_{\mathrm{ZPF}}^2}{\hbar^2} S_{FF}[\Omega] \equiv n\Gamma_{\downarrow}. \tag{2.22}$$

We could then write a simple master equation for the probability $p_n(t)$ that the oscillator is in the nth Fock state:

$$\frac{d}{dt}p_n = \left[n\Gamma_{\uparrow}p_{n-1} + (n+1)\Gamma_{\downarrow}p_{n+1}\right] - \left[n\Gamma_{\downarrow} + (n+1)\Gamma_{\uparrow}\right]p_n. \tag{2.23}$$

At this stage, the connection between the classical and quantum pictures is still murky. To connect them, use the quantum equation for p_n to derive an equation for the average oscillator energy $\langle E\rangle$. One obtains

$$\frac{d}{dt}\langle E\rangle = -\gamma\langle E\rangle + \frac{\bar{S}_{FF}[\Omega]}{2M}, \tag{2.24}$$

where:

$$\gamma = \frac{x_{\mathrm{ZPF}}^2}{\hbar^2}\left(S_{FF}[\Omega] - S_{FF}[-\Omega]\right), \tag{2.25}$$

$$\bar{S}_{FF}[\Omega] = \frac{S_{FF}[\Omega] + S_{FF}[-\Omega]}{2}. \tag{2.26}$$

We see that the quantum equation for the average energy, Eq. (2.24), has an identical form to the classical equation (Eq. (2.19)), which gives us a simple way to connect our quantum noise spectral density to quantities in the classical theory:

- The symmetrized quantum noise spectral density $\bar{S}_{FF}[\Omega]$ defined in Eq. (2.26) plays the same role as the classical noise spectral density $\mathcal{S}_{FF}[\Omega]$: it heats the oscillator the same way a classical stochastic force would.
- The asymmetric-in-frequency part of the quantum noise spectral density $S_{FF}[\Omega]$ is directly related to the damping rate γ in the classical theory. The asymmetry between absorption and emission events leads to a net energy flow between the oscillator and the noise source, analogous to what one obtains from a classical viscous damping force.

We thus see that there is a direct connection to a classical noise spectral density, and moreover the "extra information" in the asymmetry of a quantum noise spectral density also corresponds to a seemingly distinct classical quantity, a damping rate. This latter connection is not so surprising. The asymmetry of the quantum noise is a direct consequence of (here) $[\hat{F}(t), \hat{F}(t')] \neq 0$. However, this same non-commutation causes the average value of $\langle\hat{F}\rangle$ to change in response to $\hat{x}(t)$ via the interaction Hamiltonian of Eq. (2.17). Using standard quantum linear response (i.e. first-order time-dependent perturbation theory, see e.g. Ch. 6 of (Bruus and Flensberg, 2004)), one finds

$$\delta\langle\hat{F}(t)\rangle = \int_{-\infty}^{\infty} dt' \, \chi_{FF}(t-t')\langle\hat{x}(t')\rangle, \tag{2.27}$$

where the force-force susceptibility is given by the Kubo formula:

$$\chi_{FF}(t) \equiv \frac{-i}{\hbar}\theta(t)\left\langle\left[\hat{F}(t),\hat{F}(0)\right]\right\rangle. \tag{2.28}$$

From the classical Langevin equation Eq. (2.18), we see that part of $\langle\hat{F}(t)\rangle$ which is in phase with $\dot{x}$ is the damping force. This leads to the definition

$$\gamma = \frac{1}{M\Omega}\left(-\mathrm{Im}\ \chi_{FF}[\Omega]\right). \tag{2.29}$$

An explicit calculation shows that the above definition is identical to Eq. (2.25), which expresses γ in terms of the noise asymmetry. Note that in the language of many-body Green functions, $-\mathrm{Im}\ \chi_{FF}$ is referred to as a spectral function, whereas the symmetrized noise $\bar{S}_{FF}[\omega]$ is known (up to a constant) as the "Keldysh" Green function.

2.2.3 Quantum fluctuation-dissipation theorem and notion of effective temperature

We saw previously (c.f. Eq. (2.16)) that in thermal equilibrium, the negative and positive frequency noise are related by a Boltzmann factor. Using the definitions in Eqs. (2.25) and (2.26), it is then straightforward to derive the quantum version of the fluctuation-dissipation theorem:

$$\bar{S}_{FF}[\Omega] = M\gamma[\Omega]\,\hbar\Omega\coth\left(\frac{\hbar\Omega}{2k_{\mathrm{B}}T}\right) = M\gamma[\Omega]\,\hbar\Omega\left(1+2n_B[\Omega]\right). \tag{2.30}$$

For $k_{\mathrm{B}}T \gg \hbar\Omega$ this reproduces the classical result of Eq. (2.20), whereas in the opposite limit, it describes zero-point noise. In our example of an oscillator coupled to a noise source, we saw that the zero-point noise corresponds to transitions where the noise source absorbs energy– at zero temperature, emission events are not possible.

What happens if our noise source is *not* in thermal equilibrium? In that case, one could simply use Eq. (2.30) to define an effective temperature $T_{\mathrm{eff}}[\Omega]$ from the ratio of the symmetrized noise and damping. Re-writing things in terms of the quantum noise spectral density, one finds

$$k_{\mathrm{B}}T_{\mathrm{eff}}[\Omega] \equiv \hbar\Omega\left[\ln\left(\frac{S_{FF}[\Omega]}{S_{FF}[-\Omega]}\right)\right]^{-1}. \tag{2.31}$$

The effective temperature at a given frequency Ω characterizes the asymmetry between absorption and emission rates of energy $\hbar\Omega$; a large temperature indicates that these rates are almost equal, whereas a small temperature indicates that emission by the noise source is greatly suppressed compared to absorption by the source. Away from thermal equilibrium, there is no guarantee that the ratio on the RHS will be frequency-independent, and hence T_{eff} will generally have a frequency dependence.

The effective temperature of a non-equilibrium is indeed physically meaningful, especially in the case where one only probes the noise at a single, well-defined frequency. For example, in our oscillator system, only the noise at $\pm\Omega$ is important, as we are considering the limit of a very high quality factor. It is straightforward to show that the stationary energy distribution of the oscillator found by solving Eq. (2.23) is a thermal distribution evaluated at the temperature $T_{\text{eff}}[\Omega]$ defined above. The notion of an effective temperature plays an important role in so-called backaction cooling techniques (Clerk and Bennett, 2005; Blencowe *et al.*, 2005; Marquardt *et al.*, 2007; Wilson-Rae *et al.*, 2007); it also plays a significant role in the quantum theory of linear amplification, as we shall soon see.

2.3 Brief example: current noise of a quantum point contact

We have seen that we can view quantum noise either in terms of transition rates between energy eigenstates of the noise source, or in analogy to a classical noise process. An example which illustrates this dichotomy very nicely is the current noise of a quantum point contact. We sketch this result quickly in what follows; more details can be found in (Blanter and Büttiker, 2000).

A quantum point contact (QPC) is a quantum electronic conductor consisting of a narrow constriction in a two-dimensional electron gas (typically formed using gate electrodes placed above the 2DEG). The constriction is narrow enough that the transverse momentum of electrons is quantized; we will focus on the single-channel case, where only a single transverse mode is occupied. Such QPCs can be used as extremely sensitive charge sensors (Field *et al.*, 1993), and are routinely used as detectors of both quantum dot qubits (Elzerman *et al.*, 2004; Petta *et al.*, 2004), and even mechanical oscillators (Poggio *et al.*, 2008). The basic idea of the detection is that the input signal (a charge) changes the potential of the QPC, and hence changes its conductance. By monitoring the QPC current, one can thus learn about the signal.

Given its use as a detector, one is naturally very interested in current noise of a quantum point contact. To a good approximation, one can view this system as a simple one-dimensional scattering problem, with the QPC constriction acting as a scattering potential separating two ideal 1D wires (the "leads"). One can easily find the single particle scattering states, $\phi_\alpha(\varepsilon, z)$, which describe the scattering of an electron incident from the lead α (in a plane wave state) having energy ε ($\alpha = L, R$). Turning to a many-particle description, each single particle scattering state is now described by a fermionic annihilation operator $\hat{c}_\alpha(\varepsilon)$. It is straightforward to write the electronic field operator in terms of these states, and then use it to write the current operator (say in the left lead) with the result

$$\hat{I}(t) = \int d\varepsilon_1 \int d\varepsilon_2 \sum_{\alpha,\beta=L,R} A_{\alpha\beta}(\varepsilon_1, \varepsilon_2) e^{-i(\varepsilon_2-\varepsilon_1)t}\, \hat{c}^\dagger_\alpha(\varepsilon_1)\hat{c}_\beta(\varepsilon_2). \tag{2.32}$$

The matrix A here is determined by the scattering matrix of the system; an explicit form is given in, e.g., (Blanter and Büttiker, 2000). For finite temperature and voltage, the QPC "leads" are taken to be in equilibrium, but at different chemical potentials, with $\mu_L - \mu_R = eV$ giving the applied bias voltage. Using Eq. (2.13) we see that the current noise at frequency ω naturally corresponds to transitions between scattering states whose energy differs by $\hbar\omega$. The probability for a given scattering state incident from lead α to be occupied is just given by a Fermi function evaluated at the chemical potential μ_α. At zero temperature the Fermi function becomes a step function, and all states are either empty, or are occupied with probability one. In this case, the only possible transitions at $\omega = 0$ involve states in the energy window $\mu_R < \varepsilon < \mu_L$, with the relevant transitions being the removal of a left scattering state, and the creation of a right scattering state at the same energy.

We thus obtain a picture of the current noise which appears to be explicitly quantum mechanical, involving transitions between energy eigenstates of the detector (here, just scattering states). Remarkably, a simple classical interpretation in terms of a classical noise process is also possible. Note first that at zero temperature (and ignoring any energy-dependence of the scattering), the average current through the QPC obtained from Eq. (2.32) is simply given by the Landauer-Büttiker result

$$\frac{1}{e}\langle \hat{I} \rangle = \frac{eV}{h}\mathcal{T}, \tag{2.33}$$

where $0 \leq \mathcal{T} \leq 1$ is the transmission probability of the QPC. One also finds from Eq. (2.32) that the zero frequency current noise is given by

$$\frac{1}{e^2}\bar{S}_{II}[0] = \frac{eV}{h}\mathcal{T}\left(1 - \mathcal{T}\right). \tag{2.34}$$

These results correspond to a simple classical noise process, where electrons are launched one at a time towards the scattering potential at a rate eV/h. In each event, the electron is either transmitted (probability $\mathcal{T}$) or reflected (probability $1-\mathcal{T}$). We thus have a binomial process, similar to flipping an unevenly-weighted coin. The average current just corresponds to the average number of electrons that are transmitted, while the noise corresponds to the variance of the binomial process. If we imagine integrating $\hat{I}(t)$ up from $t = 0$ (i.e. counting electrons), then in the long-time limit, we have a variance

$$\frac{1}{e^2}\left\langle\left\langle \left(\int_0^t dt'\, \hat{I}(t')\right)^2 \right\rangle\right\rangle = \frac{eVt}{h}\mathcal{T}\left(1 - \mathcal{T}\right) \equiv N_{\mathrm{A}}\mathcal{T}\left(1 - \mathcal{T}\right). \tag{2.35}$$

Here, we can view N_{A} as the number of attempts at getting an electron through the QPC. We thus have a concrete example showing the two complimentary ways that one can look at quantum noise in general (i.e. in terms of transitions between eigenstates, versus in analogy to a classical noise process).

2.4 Heisenberg inequality on detector quantum noise

2.4.1 Generic two-port linear response detector

Having discussed two of the ways quantum noise spectral densities differ from their classical counterparts (zero-point noise, frequency asymmetry), we now turn to the

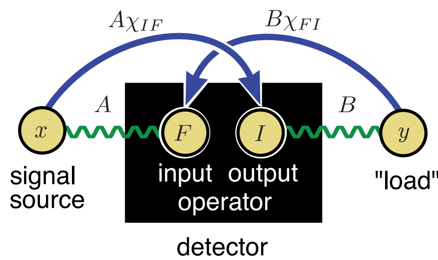

Fig. 2.1 (Color online) Schematic of a generic linear response detector.

third distinguishing feature: there are purely quantum constraints on the noise properties of any system capable of acting as a detector or amplifier. We will be interested in the generic two-port detector sketched in Fig. 2.1. The detector has an input port characterized by an operator $\hat{F}$: this is the detector quantity which couples to the system we wish to measure. Similarly, the output port is characterized by an operator $\hat{I}$: this is the detector quantity that we will readout to learn about the system coupled to the input. For example, for a QPC detector coupled to a double dot, the state of the qubit σ_z changes the potential of electrons in the QPC. The operator $\hat{F}$ will thus involve the charge-density operator of the QPC electrons. In contrast, the quantity that is actually measured is the QPC current; hence, $\hat{I}$ will be the QPC current operator.

We will be interested almost exclusively in detector-signal couplings weak enough that one can use linear-response to describe how $\hat{I}$ changes in response to the signal. For example, if we couple an input signal $\hat{z}$ to our detector via an interaction Hamiltonian

$$\hat{H}_{\text{int}} = \hat{z} \cdot \hat{F}, \tag{2.36}$$

linear response tells us that the change in the detector output will be given by:

$$\delta \langle \hat{I}(t) \rangle = \int_{-\infty}^{\infty} dt' \chi_{IF}(t-t') \langle \hat{z}(t') \rangle, \tag{2.37}$$

$$\chi_{IF}(t) = -\frac{i}{\hbar} \theta(t) \left\langle [\hat{I}(t), \hat{F}(0)] \right\rangle. \tag{2.38}$$

This is completely analogous to the way we discussed damping, c.f. Eq. (2.29). As is standard in linear-response, the expectation values above are all with respect to the state of the system (signal plus detector) at zero coupling (i.e. $\hat{H}_{\text{int}} = 0$). Also, without loss of generality, we will assume that both $\langle \hat{I} \rangle$ and $\langle \hat{F} \rangle$ are zero in the absence of any coupling to the input signal.

Even on a classical level, any noise in the input and output ports will limit our ability to make measurements with the detector. Quantum mechanically, we have seen that it is the symmetrized quantum spectral densities that play a role analogous to classical noise spectral densities. We will thus be interested in the quantities $\bar{S}_{II}[\omega]$ and $\bar{S}_{FF}[\omega]$. Given our interest in weak detector-signal couplings, it will be sufficient to characterize the detector noise at zero-coupling to the detector (though we will go beyond this assumption in our discussion of qubit detection).

In addition to $\bar{S}_{II}, \bar{S}_{FF}$, we will also have to contend with the fact that the noise in $\hat{I}$ and $\hat{F}$ may be correlated. Classically, we would describe such correlations via a correlation spectral density $\mathcal{S}_{IF}[\omega]$:

$$\mathcal{S}_{IF}[\omega] \equiv \lim_{T\to\infty} \langle I_T[\omega] \left(F_T[\omega]\right)^* \rangle = \int_{-\infty}^{\infty} dt \, \langle I(t)F(0)\rangle e^{i\omega t}, \tag{2.39}$$

where the Fourier transforms $I_T[\omega]$ and $F_T[\omega]$ are defined analogously to Eq. (2.2). Not surprisingly, such classical correlations correspond to a symmetrized quantum noise spectral density

$$\bar{S}_{IF}[\omega] \equiv \frac{1}{2} \int_{-\infty}^{\infty} dt \, \langle \{\hat{I}(t), \hat{F}(0)\}\rangle e^{i\omega t}. \tag{2.40}$$

Note that the classical correlation density $\mathcal{S}_{IF}[\omega]$ is generally complex, and is only guaranteed to be real at $\omega = 0$; the same is true of $\bar{S}_{IF}[\omega]$.

Finally, we normally are only concerned about how large the output noise is compared to the magnitude of the "amplified" input signal at the output (i.e. Eq. (2.37)). It is thus common to think of the output noise at a given frequency $\delta I_T[\omega]$ as an equivalent fluctuation of the signal $\delta z_{\mathrm{imp}}[\omega] \equiv \delta I_T[\omega]/\chi_{IF}[\omega]$. We thus define the imprecision noise spectral density and imprecision-backaction correlation density as:

$$\bar{S}_{zz}[\omega] \equiv \frac{\bar{S}_{II}[\omega]}{|\chi_{IF}[\omega]|^2}, \quad \bar{S}_{zF}[\omega] \equiv \frac{\bar{S}_{IF}[\omega]}{\chi_{IF}[\omega]}. \tag{2.41}$$

2.4.2 Motivation and derivation of noise constraint

We can now ask what sort of constraints exist on the detector noise. In almost all relevant cases, our detector will be some sort of driven quantum system, and hence will not be in thermal equilibrium. As a result, any meaningful constraint should not rely on having a thermal equilibrium state. Classically, all we can say is that the correlations in the noise cannot be bigger than the noise itself. This constraint takes the form of a a Schwartz inequality, yielding

$$\mathcal{S}_{zz}[\omega]\mathcal{S}_{FF}[\omega] \geq \left|\mathcal{S}_{zF}[\omega]\right|^2. \tag{2.42}$$

Equality here implies a perfect correlation, i.e. $I_T[\omega] \propto F_T[\omega]$.

Quantum mechanically, additional constraints will emerge. Heuristically, this can be expected by making an analogy to the example of the Heisenberg microscope. In that example, one finds that there is a tradeoff between the imprecision of the measurement (i.e. the position resolution) and the backaction of the measurement (i.e. the momentum kick delivered to the particle). In our detector, noise in $\hat{I}$ will correspond to the imprecision of the measurement (i.e. the bigger this noise, the harder it will be to resolve the signal described by Eq. (2.37)). Similarly, noise in $\hat{F}$ is the backaction: as we already saw, by virtue of the detector-signal coupling, $\hat{F}$ acts as a noisy force on the measured quantity $\hat{z}$. Making the analogy to Eq. (1.1) for the Heisenberg microscope, we thus might expect a bound on the product of $\bar{S}_{zz}\bar{S}_{FF}$.

Alternatively, we see from Eq. (2.38) that for our detector to have any response at all, $\hat{I}(t)$ and $\hat{F}(t')$ cannot commute for all times. Quantum mechanically, we know that uncertainty relations apply any time we have non-commuting observables; here things are somewhat different, as the non-commutation is between Heisenberg-picture operators at different times. Nonetheless, we can still use the standard derivation of an uncertainty relation to obtain a useful constraint. Recall that for two non-commuting observables $\hat{A}$ and $\hat{B}$, the full Heisenberg inequality is (see, e.g. (Gottfried, 1966))

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left\langle\left\{\hat{A},\hat{B}\right\}\right\rangle^2 + \frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|^2. \tag{2.43}$$

Here we have assumed $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$. We now take $\hat{A}$ and $\hat{B}$ to be cosine-transforms of $\hat{I}$ and $\hat{F}$, respectively, over a finite time-interval T:

$$\hat{A} \equiv \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} dt\, \cos(\omega t + \delta)\, \hat{I}(t), \quad \hat{B} \equiv \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} dt\, \cos(\omega t)\, \hat{F}(t). \tag{2.44}$$

Note that we have phase shifted the transform of $\hat{I}$ relative to that of $\hat{F}$ by a phase δ. In the limit $T \to \infty$ we find

$$\bar{S}_{zz}[\omega]\bar{S}_{FF}[\omega] \geq \left[\mathrm{Re}\ \left(e^{i\delta}\bar{S}_{zF}[\omega]\right)\right]^2 + \frac{\hbar^2}{4}\left[\mathrm{Re}\ e^{i\delta}\left(1 - \frac{(\chi_{FI}[\omega])^*}{\chi_{IF}[\omega]}\right)\right]^2. \tag{2.45}$$

We have introduced here a new susceptibility $\chi_{FI}[\omega]$, which describes the reverse response coefficient or reverse gain of our detector. This is the response coefficient relevant if we used our detector in reverse: couple the input signal $\hat{z}$ to $\hat{I}$, and see how $\langle\hat{F}\rangle$ changes. A linear response relation analogous to Eq. (2.37) would then apply, with $F \leftrightarrow I$ everywhere. We define the ratio of the detector response coefficients to be

$$r[\omega] = \frac{(\chi_{FI}[\omega])^*}{\chi_{IF}[\omega]}. \tag{2.46}$$

If we now maximize the RHS of Eq. (2.45) over all values of δ, we are left with the optimal bound

$$\bar{S}_{zz}[\omega]\bar{S}_{FF}[\omega] - \left|\bar{S}_{zF}[\omega]\right|^2 \geq \frac{\hbar^2}{4}\left|1 - r[\omega]\right|^2\left(1 + \Delta\left[\frac{2\bar{S}_{zF}[\omega]}{\hbar(1 - r[\omega])}\right]\right), \tag{2.47}$$

where

$$\Delta[y] = \frac{\left|1 + y^2\right| - \left(1 + |y|^2\right)}{2}, \tag{2.48}$$

Note that for any complex number y, $1 + \Delta[y] > 0$. Related noise constraints on linear-response detectors are presented in (Braginsky and Khalili, 1996) and (Averin, 2003).

We see that applying the uncertainty principle to our detector has given us a rigorous constraint on the detector's noise which is stronger than the simple classical bound of Eq. (2.42) on its correlations. This extra quantum constraint vanishes if our detector has completely symmetric response coefficients, that is $r[\omega] = 1$. For simplicity, consider first the $\omega \to 0$ limit, where all noise spectral densities and susceptibilities are real, and hence the term involving $\Delta[y]$ vanishes. For a non-symmetric detector, the extra quantum term on the RHS of Eq. (2.45) then implies:

- The product of the imprecision noise $\bar{S}_{zz}$ and backaction noise $\bar{S}_{FF}$ cannot be zero. The magnitude of both kinds of fluctuations must be non-zero.
- Moreover, these fluctuations cannot be perfectly correlated with one another: we cannot have $\left(\bar{S}_{zF}\right)^2 = \bar{S}_{zz}\bar{S}_{FF}$.

The presence of these extra quantum constraints on noise will lead to fundamental quantum limits on various things we might try to do with our detector; this will be the focus of the remainder of these lectures.

2.4.3 Comments on the reverse gain of a detector

Before moving on, it is worth commenting more on the reverse-gain χ_{FI}: both its meaning and its role in the quantum noise inequality have been the subject of some confusion. Several points are worth noting:

- If our detector is in thermal equilibrium and in a time-reversal symmetric state, then the relationship between χ_{IF} and χ_{FI} is constrained by Onsager reciprocity relations (see, e.g., (Pathria, 1996) for an elementary discussion). One has $\chi_{IF}[\omega] = \pm\chi_{FI}[\omega]^*$ where the $+$ $(-)$ sign corresponds to the case where both $\hat{F}$ and $\hat{I}$ have the same (opposite) parity under time-reversal. For example, in a QPC detector, $\hat{F}$ is a charge and hence even under time-reversal, where $\hat{I}$ is a current and odd under time-reversal; one thus has $\chi_{IF}[\omega] = -\chi_{FI}[\omega]^*$. It follows that in thermal equilibrium, if one has forward response, then one must necessarily also have reverse response.
- In general, it is highly undesirable to have non-zero reverse gain. To make a measurement of the output operator $\hat{I}$, we must necessarily couple to it in some manner. If $\chi_{FI} \neq 0$, the noise associated with this coupling could in turn lead to additional back-action noise in the operator $\hat{F}$, above and beyond the intrinsic fluctuations described by $\bar{S}_{FF}$. This is clearly something to be avoided. Thus, the ideal situation is to have $\chi_{FI} = 0$, implying a high asymmetry between the input and output of the detector, and requiring the detector to be in a state far from thermodynamic equilibrium.

2.4.4 Ideal quantum noise

We can now define in a general and sensible manner what it means for a detector to possess "ideal" quantum noise at a frequency ω: we require that the detector optimizes the fundamental quantum noise inequality, i.e. Eq. (2.47) holds as an equality. We will see that having such ideal quantum noise properties is a pre-requisite for achieving various quantum limits on measurement. It also places tight constraints on the property of our detector. One can show having ideal quantum noise necessarily implies (Clerk *et al.*, 2010):

- In a certain restricted sense, the operators $\hat{I}$ and $\hat{F}$ must be proportional to one another. More formally, $\bar{S}_{II}[\omega]$ and $\bar{S}_{FF}[\omega]$ can be written as sums transitions between detector energy eigenstates $|i\rangle$ and $|f\rangle$ whose energy differs by $\pm\hbar\omega$ (c.f. Eq. (2.13)). To have quantum ideal noise, one needs that for each such contributing transitions, the ratio of the matrix elements $\langle f|\hat{I}|i\rangle/\langle f|\hat{F}|i\rangle$ is the same.
- As long as $|r[\omega]| \neq 1$, the detector *cannot* be in a thermal equilibrium state.

- For a general non-equilibrium system, the effective temperature defined using the asymmetry of the quantum noise spectral density $S_{II}[\omega]$ (c.f. Eq. (2.31)) need not be the same as that defined using $S_{FF}[\omega]$. However, having quantum-ideal noise necessarily implies that both these effective temperatures are the same. Paradoxically, while such a detector cannot be in equilibrium, its effective temperature is in some sense more universal than that of an arbitrary non-equilibrium system.

At this stage, the true meaning of the quantum noise inequality may seem quite opaque. It may also seem that there is no hope of understanding in general what one needs to do to achieve ideal quantum noise. However, by considering concrete examples, we will gain insights into both these issues.

3

Quantum limit on QND qubit detection

3.1 Measurement rate and dephasing rate

Armed now with a basic understanding of quantum noise and the Heisenberg bounds which constrain it, we can finally consider doing something useful with our generic linear response detector. To that end, we consider a qubit whose Hamiltonian is

$$\hat{H}_{\mathrm{qb}} = \frac{\hbar\Omega}{2}\hat{\sigma}_z. \tag{3.1}$$

Suppose we want to measure whether the qubit is in its ground or excited state. We start by coupling its $\hat{\sigma}_z$ operator to the input of our detector:

$$\hat{H}_{\mathrm{int}} = A\hat{\sigma}_z\hat{F}. \tag{3.2}$$

By virtue of Eq. (2.37), the two different qubit eigenstates $|\uparrow\rangle$, $|\downarrow\rangle$ will lead to two different average values of the detector $\langle\hat{I}\rangle$; thus, by looking at the detector output, we can measure the value of σ_z.

Note crucially that $[\hat{H}_{\mathrm{int}}, \hat{H}_{\mathrm{qb}}] = 0$. As such, $\langle\hat{\sigma}_z\rangle$ is a constant of the motion even when the qubit is coupled to the detector: if the qubit starts in an energy eigenstate, it will remain in that state. Detection schemes where the coupling commutes with the system Hamiltonian are known as being "quantum non-demolition" (QND). On practical level, this can be extremely useful, as one can leave the measurement on for a long time (or make multiple measurements) to improve the precision without worrying about the measurement process altering the value of the measured observable.

Because of the intrinsic noise in the output of our detector (described by $\bar{S}_{II}$), it will take some time before we can tell whether the qubit is up or down. We only gradually obtain information about the qubit state, and can rigorously define a rate to characterize this process, the so-called measurement rate. Imagine we turn the measurement on at $t = 0$, and start to integrate up the output $I(t)$ of our detector:

$$\hat{m}(t) = \int_0^t dt'\hat{I}(t'). \tag{3.3}$$

The probability distribution of the integrated output $\hat{m}(t)$ will depend on the state of the qubit; for long times, we may approximate the distribution corresponding to each qubit state as being gaussian. Noting that we have chosen $\hat{I}$ so that its expectation value vanishes at zero coupling, the average value of $\langle \hat{m}(t) \rangle$ corresponding to each qubit state is:

$$\langle \hat{m}(t) \rangle_\uparrow = A\chi_{IF}[0]t, \quad \langle \hat{m}(t) \rangle_\downarrow = -A\chi_{IF}[0]t. \tag{3.4}$$

Note that we are assuming integration times t much longer than any internal detector timescale, and thus only the zero-frequency response coefficient χ_{IF} appears. Taking the long time limit here is consistent with our assumption of a weak detector-qubit coupling: it will take a long time before we get information on the qubit state.

Next, let's consider the uncertainty in the quantity m as described by its variance $\langle\langle m^2 \rangle\rangle \equiv \langle \hat{m}^2 \rangle - \langle \hat{m} \rangle^2$. For weak coupling, we can ignore the fact that the variance will have a small dependence on the qubit state, as this will only lead to higher-order-in-A corrections to our expression for the measurement rate. We thus have

$$\langle \hat{m}^2(t) \rangle \equiv \int_0^t dt_1 \int_0^t dt_2 \, \langle \hat{I}(t_1)\hat{I}(t_2) \rangle \to \bar{S}_{II}[0]t. \tag{3.5}$$

We have again taken the limit where t is much larger than the correlation time of the detector noise, and hence the variance is completely determined by the zero-frequency output noise.

We can now define the measurement rate by how quickly the resolving power of the measurement grows:[1]

$$\frac{1}{4} \frac{\left[\langle \hat{m}(t) \rangle_\uparrow - \langle \hat{m}(t) \rangle_\downarrow\right]^2}{\langle\langle \hat{m}^2(t) \rangle\rangle_\uparrow + \langle\langle \hat{m}^2(t) \rangle\rangle_\downarrow} \equiv \Gamma_{\mathrm{meas}} t. \tag{3.6}$$

This yields

$$\Gamma_{\mathrm{meas}} = \frac{A^2 \left(\chi_{IF}\right)^2}{2\bar{S}_{II}}. \tag{3.7}$$

We can think of $1/\Gamma_{\mathrm{meas}}$ as a measurement time, i.e. the amount of time we have to wait before we can reliably determine whether the qubit is up or down (above the intrinsic noise in the detector output).

Having characterized the imprecision of the measurement, we now turn to its backaction. At first glance, one might think the fact that we have a QND setup implies the complete absence of measurement backaction. This is not true. We are making a measurement of σ_z, and hence there must be a backaction disturbance of the conjugate quantities σ_x, σ_y. More explicitly, if we start the qubit out in a superposition of energy eigenstates, then the phase information of this superposition will be lost gradually in time due to the backaction of the measurement.

[1] The strange looking factor of 1/4 here is purely chosen for convenience, as it will let us formulate the quantum limit in a way that involves no numerical prefactors. Interestingly enough, the prefactor can be rigorously justified if one uses the accessible information (a standard information-theoretic measure) to quantify the difference between the output distributions; the measurement rate as defined is precisely the rate of growth of the accessible information (Clerk *et al.*, 2003).

To describe this backaction effect, note first that we can incorporate the coupling into the qubit's Hamiltonian as

$$\hat{H}_{\mathrm{qb}} + \hat{H}_{\mathrm{int}} = \left(\frac{\hbar\Omega}{2} + A\hat{F} \right) \hat{\sigma}_z. \tag{3.8}$$

Thus, from the qubit's point of view, the coupling to the detector means that its splitting frequency has a randomly fluctuating part described by $\Delta\Omega = 2A\hat{F}/\hbar$. This effective frequency fluctuation will cause a diffusion of the qubit's phase in the long time limit according to

$$\left\langle e^{-i\varphi} \right\rangle = \left\langle e^{-i\int_0^t d\tau\, \Delta\Omega(\tau)} \right\rangle. \tag{3.9}$$

For weak coupling the dephasing rate is slow and thus we are interested in long times t. In this limit the integral is a sum of a large number of statistically independent terms and thus we can take the accumulated phase to be Gaussian distributed. Using the cumulant expansion we then obtain

$$\begin{aligned} \left\langle e^{-i\varphi} \right\rangle &= \exp\left(-\frac{1}{2} \left\langle \left[\int_0^t d\tau\, \Delta\Omega(\tau) \right]^2 \right\rangle \right) \\ &= \exp\left(-\frac{2A^2}{\hbar^2} \bar{S}_{FF}[0] t \right) \equiv \exp\left(-\Gamma_\varphi t \right). \end{aligned} \tag{3.10}$$

We have again taken the long time limit, which means that the only the zero-frequency backaction noise spectral density enters. Eq. (3.10) yields the dephasing rate

$$\Gamma_\varphi = \frac{2A^2}{\hbar^2} \bar{S}_{FF}[0]. \tag{3.11}$$

3.2 Efficiency ratio

On a completely heuristic level, we can easily argue that the measurement and dephasing rates of our setup should be related. Imagine a simple case where at $t = 0$ the qubit is in a superposition state, and the detector is in some pure state $|D_0\rangle$:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + e^{i\varphi_0} |\downarrow\rangle \right) \otimes |D_0\rangle. \tag{3.12}$$

At some later time, due to the qubit-detector interaction, the qubit can become entangled with the detector, and we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |D_\uparrow(t)\rangle + e^{i\varphi_0} |\downarrow\rangle \otimes |D_\downarrow(t)\rangle \right), \tag{3.13}$$

where the two detector states are not necessarily equal: $|D_\uparrow(t)\rangle \neq |D_\downarrow(t)\rangle$.

To see if the qubit has dephased or not, consider an off-diagonal element of its reduced density matrix:

$$\rho_{\downarrow\uparrow}(t) \equiv \langle \downarrow | \hat{\rho}(t) | \uparrow \rangle = \frac{e^{i\varphi_0}}{2} \langle D_\uparrow(t) | D_\downarrow(t) \rangle. \tag{3.14}$$

At $t = 0$, $|D_\uparrow(0)\rangle = |D_\downarrow(0)\rangle = |D_0\rangle$, and the off-diagonal density matrix element $\rho_{\downarrow\uparrow}(0)$ just tells us the initial qubit phase. As t increases from 0, $|D_\uparrow(t)\rangle$ and $|D_\downarrow(t)\rangle$ will in general be different, causing the magnitude of $\rho_{\downarrow\uparrow}(t)$ to decay with time. Comparing against Eq. (3.10), we would thus associate backaction dephasing with the fact that the two detector states $|D_\uparrow(t)\rangle$, $|D_\downarrow(t)\rangle$ have an overlap of magnitude less than one.

In contrast, the measurement rate in Eq. (3.7) does not directly involve the overlap of the two detector states $|D_\uparrow(t)\rangle$, $|D_\downarrow(t)\rangle$. Rather, it involves how different the distributions of $\hat{m}$ are in these two states. Clearly, if the two states $|D_\uparrow\rangle$, $|D_\downarrow\rangle$ can be distinguished by looking at m, then they must have an overlap < 1: *measurement implies dephasing*. The converse is not true: the two detector states could be orthogonal because the qubit has become entangled with extraneous detector degrees of freedom, without these states yielding different distributions of m. Hence, *dephasing does not imply measurement*.

Putting this together, one roughly expects that the measurement rate should be bounded by the dephasing rate. We can test this expectation by using the linear-response expressions derived in the previous section, and making use of the quantum noise inequality of Eq. (2.47). If we assume the ideal case of zero reverse gain in our detector, we find

$$\eta \equiv \frac{\Gamma_{\text{meas}}}{\Gamma_\varphi} = \frac{\hbar^2/4}{\bar{S}_{zz}\bar{S}_{FF}} \leq 1. \tag{3.15}$$

Thus, in the absence of any detector reverse gain, we obtain the expected result: the dephasing rate must be at least as large as the measurement rate. This is the quantum limit on QND qubit detection (Devoret and Schoelkopf, 2000; Averin, 2000; Korotkov and Averin, 2001; Makhlin *et al.*, 2001; Clerk *et al.*, 2003). This derivation does more than prove the bound, it also indicates what we need to do to reach it. We need both:

1. A detector with quantum ideal noise at zero frequency, that is must saturate the inequality of Eq. (2.47) at $\omega = 0$.
2. There must be *no* backaction - imprecision noise correlations at zero frequency: $\bar{S}_{zF}$ must be zero

3.2.1 Violating the quantum limit with reverse gain?

Despite the intuitive reasonableness of the above quantum limit on QND qubit detection, there would seem to be a troubling loophole in the case where our detector has a non-zero reverse gain χ_{FI}. In this case, the RHS of our so-called "quantum limit" is now $(1 - r[0])^2$ (where $r[0]$ is the ratio of the reverse to forward gains at zero frequency, c.f. Eq. (2.46)), and can be made arbitrarily small by having our detectors forward and reverse responses be symmetric. One is tempted to conclude that there is in fact no quantum limit on QND qubit detection. This is of course an invalid inference: as discussed, $\chi_{FI} \neq 0$ implies that we must necessarily consider the effects of extra noise injected into detector's output port when one measures $\hat{I}$, as the reverse gain will bring this noise back to the qubit, causing extra dephasing. The result is that one can do no better than $\eta = 1$. To see this explicitly, consider the extreme case $\chi_{IF} = \chi_{FI}$ and $\bar{S}_{II} = \bar{S}_{FF} = 0$, and suppose we use a second detector to read-out the output $\hat{I}$ of the first detector. This second detector has input and output operators $\hat{F}_2$, $\hat{I}_2$; we also take it to have a vanishing reverse gain, so that we do not have to also worry about how its output is read-out. Coupling the detectors linearly in the standard way (i.e. $H_{\text{int},2} = \hat{I}\hat{F}_2$), the overall gain of the two detectors in series is $\chi_{I_2F_2} \cdot \chi_{IF}$, while the back-action driving the qubit dephasing is described by the spectral density $(\chi_{FI})^2 S_{F_2F_2}$. Using the fact that our second detector must itself satisfy the quantum noise inequality, we have

$$\left[(\chi_{FI})^2 \bar{S}_{F_2F_2}\right] \bar{S}_{I_2I_2} \geq \frac{\hbar^2}{4} \left(\chi_{I_2F_2} \cdot \chi_{IF}\right)^2. \tag{3.16}$$

Thus, the overall chain of detectors satisfies the usual, zero-reverse gain quantum noise inequality, implying that we will still have $\eta \leq 1$.

3.3 Example: QPC detector

Let's consider again the single-channel quantum point contact detector of Sec. 2.3, and imagine that we connect it to a single-electron, double quantum dot. The single electron can be in either the left or the right dot; these will correspond to the σ_z eigenstates of the effective qubit formed by the dot. If we assume that interdot tunnelling has been switched off, then these two states are also energy eigenstates of the qubit. Finally, these two states will lead to to different electrostatic potentials for the QPC electrons: we thus have a coupling of the form given in Eq. (2.36), where the input $\hat{F}$ operator is actually a charge in the QPC.

A rigorous treatment of the measurement properties of a QPC detector is given in (Clerk *et al.*, 2003; Pilgram and Büttiker, 2002; Young and Clerk, 2010). Here, we provide a more heuristic treatment which brings out the main aspects of the physics, and also helps motivate the crucial connection between the quantum limit on QND qubit detection, the quantum noise constraint of Eq. (2.47), and the principle of "no wasted information".

Recall first our results for the current and current noise of a QPC detector (c.f. Eqs. (2.33) and (2.35)); both depend on $\mathcal{T}$, the probability of electron transmission through the QPC. Including the coupling to the qubit, each qubit eigenstate will correspond to two different effective QPC potentials, and hence two different transmission coefficients $\mathcal{T}_\uparrow, \mathcal{T}_\downarrow$:

$$\mathcal{T}_\uparrow \equiv \mathcal{T}_0 + \Delta\mathcal{T}, \quad \mathcal{T}_\downarrow \equiv \mathcal{T}_0 - \Delta\mathcal{T}. \tag{3.17}$$

Using the above equations for $\langle \hat{I} \rangle$ and $\bar{S}_{II}$, Eq. (3.7) for the measurement rate immediately yields

$$\Gamma_{\text{meas}} = \frac{1}{2} \frac{(\Delta\mathcal{T})^2}{\mathcal{T}_0(1-\mathcal{T}_0)} \frac{eV}{h}. \tag{3.18}$$

Turning to the dephasing rate, the backaction charge fluctuations $\bar{S}_{FF}$ can be calculated using scattering theory, see (Clerk *et al.*, 2003). We instead take a more heuristic approach that yields the correct answer and provides us the general insight we are after. Let's describe the transmitted charge m through the QPC with an approximate wavefunction; further, let's ignore the discreteness of charge, and treat m

to be continuous. Thus, if the qubit was initially in state α, the transmitted charge through the QPC at time t might reasonably be described by a wavefunction

$$|\psi_{\mathrm{QPC},\alpha}(t)\rangle = \int dm\, \phi_\alpha(m)|m\rangle. \tag{3.19}$$

We pick $\phi_\alpha(m)$ to yield the expected (Gaussian) probability distribution of m,

$$|\phi_\alpha(m)|^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(m-\bar{m}_\alpha)^2}{2\sigma^2}\right), \tag{3.20}$$

where the mean and variance match what we already calculated:

$$\bar{m}_\alpha = \frac{eVt}{h}\left(\mathcal{T}_0 \pm \Delta\mathcal{T}\right), \quad \sigma = \frac{1}{e^2}\bar{S}_{II}t = \frac{eVt}{h}\mathcal{T}_0\left(1-\mathcal{T}_0\right). \tag{3.21}$$

We still have to worry about the phase of our phenomenological QPC wavefunction. Recall that our description of the QPC is based on a simple scattering picture. An incident electron is either transmitted with an amplitude $\sqrt{\mathcal{T}}e^{i\theta_t}$, or reflected with an amplitude $\sqrt{1-\mathcal{T}}e^{i\theta_r}$. Here, θ_t and θ_r are phases in the scattering matrix. If we use the state where all electrons are reflected as our phase reference, we see that each transmission event is associated with a net phase shift

$$\exp\left(i(\theta_t - \theta_r)\right) \equiv \exp\left(i\theta\right). \tag{3.22}$$

Further, in the same way that the two states of the qubit can change the transmission probability $\mathcal{T}$, they could also cause the value of this phase difference to change. We thus write:

$$\theta_\uparrow = \theta_0 + \Delta\theta, \quad \theta_\downarrow = \theta_0 - \Delta\theta. \tag{3.23}$$

Based on this picture, it is reasonable to write our final heuristic wavefunction in the form

$$\phi_\alpha(m) = |\phi_\alpha(m)|e^{im\theta_\alpha}. \tag{3.24}$$

We are now in a position to calculate the backaction dephasing rate of the qubit. Using Eq. (3.14), we see that this is just determined by the overlap of the two QPC states:

$$\exp(-\Gamma_\varphi t) \equiv \left|\langle\psi_{\mathrm{QPC},\uparrow}|\psi_{\mathrm{QPC},\downarrow}\rangle\right|. \tag{3.25}$$

We can easily calculate the required overlap, as it amounts to a simple Gaussian integration. We find

$$\Gamma_\varphi = \Gamma_{\mathrm{meas}} + 2\left(\Delta\theta\right)^2 \mathcal{T}_0\left(1-\mathcal{T}_0\right)\frac{eV}{h} \equiv \Gamma_{\mathrm{meas}} + \Gamma_{\mathrm{meas},\theta}. \tag{3.26}$$

where Γ_{meas} is just the measurement rate given in Eq. (3.18). We note that this expressions matches exactly what is found from a rigorous calculation of the backaction noise $\bar{S}_{FF}$ (Pilgram and Büttiker, 2002; Clerk, Girvin and Stone, 2003). We see that if $\Delta\theta \neq 0$, the QPC misses the quantum limit on QND qubit detection: the dephasing rate is larger than the measurement rate. The extra term in the dephasing rate can be directly interpreted as the measurement rate of an experiment where one tried to determine the qubit state by interfering transmitted and reflected beams of electrons. This "phase" contribution to the dephasing rate has even been measured in experiments using quantum hall edge states (Sprinzak *et al.*, 2000).

Several comments are in order:

- We see that a failure to reach the QND quantum limit corresponds to the existence of "wasted information": there are other quantities besides $\hat{I}$ that one could measure to learn about the state of the qubit. The corollary is that to reach the quantum limit, there should be no wasted information: there should be no other degrees of freedom in the detector that could provide more information on the qubit state besides that available in $\hat{I}$. This idea is of course more general than just this example, and provides a powerful way of assessing whether a given system will reach the quantum limit.
- The same reasoning applies to the noise properties of the detector: a failure to optimize the quantum noise inequality of Eq. (2.47) is in general associated with "wasted information" in the detector. Many more examples of this are given in (Clerk *et al.*, 2010).

As a final comment, we note that reaching the quantum limit on QND qubit detection not only requires having a detector with "ideal" quantum noise, but in addition, there must be no backaction-imprecision noise correlations. Such correlated backaction noise is always in excess of the absolute minimum value of $\bar{S}_{FF}$ required by Eq. (2.47). As we will see, in non-QND measurements one can make use of these correlations, and in some cases one even requires their presence to reach the quantum limit on the measurement. In the QND case however backaction is irrelevant to what shows up in the output of the detector, and hence one cannot make use of any backaction-imprecision correlation. As such, the correlated backaction also represents a kind of wasted information.

It is interesting to note that in the QPC example, the "phase" contribution to the backaction noise is in fact perfectly correlated with the imprecision current noise. Consider the simple case where $\Delta T = 0$, and there is only "phase information" on the state of the qubit. If the qubit is initially in a superposition state with a phase ϕ_0 (c.f. Eq. (3.12)), then at time t, it follows from Eqs. (3.14),(3.24) that its off-diagonal density matrix element will be given by

$$\langle e^{-i\varphi}\rangle = e^{-i\varphi_0}\int dm\,|\phi(m)|^2\,e^{-2im\Delta\theta} = e^{-i\varphi_0}\int dm\,p(m)e^{-2im\Delta\theta}. \tag{3.27}$$

Thus, if the value of m was definite, the qubit would pick up a deterministic phase shift $2\Delta\theta m$; however, as m fluctuates, one gets a random phase shift and hence dephasing. Crucially though, this random phase shift (i.e. backaction noise) is correlated with the fluctuations of m (i.e. the imprecision current noise). In the simple limit $\Delta T \to 0$ considered here, the QPC saturates the quantum limit of Eq. (2.47), but with zero gain: $\chi_{IF} = 0$.

3.4 Significance of the quantum limit on QND qubit detection

At this stage, one might legitimately wonder why anyone would care about this quantum limit on qubit measurement. If our only goal is to determine whether the qubit is up or down, why should we care about whether the qubit is dephased as slowly as is allowed by quantum mechanics? This would seem to have no bearing on our ability to make a measurement.

The full answer to this question involves the world of conditional measurement: what happens to the qubit in a *single* run of the experiment? More concretely, in a given run of the experiment, one obtains a specific, noisy time-trace of $I(t)$. Given this time trace, what can one say about the qubit? Such knowledge is of course crucial if one wishes to use the measurement record in a feedback protocol to control the qubit state. The QND qubit quantum limit plays a crucial role here: if the detector reaches the quantum limit, then in a *particular* run of the experiment, there is no measurement-induced qubit dephasing (see, e.g., (Korotkov, 1999)). Rather, the qubit's phase undergoes a seemingly random evolution which is in fact correlated with the noise in the detector output, $I(t)$. This phase evolution only looks like dephasing when one does not have access to the measurement record. These fascinating ideas will be treated by other lectures in this school.

3.5 QND quantum limit beyond linear response

What if the qubit-detector coupling A is not so small to allow the neglect of higher-order contributions to the dephasing and measurement rates? We can still formulate a quantum limit on the backaction dephasing rate, by saying that it is bounded below by its value in the most ideal case. The most ideal case corresponds to the situation in Eq. (3.13), where each qubit state leads to a different detector pure state. Further, the most ideal situation is where the overlap of these two states is *completely* determined by the probability distribution of the integrated detector output m (i.e. like our heuristic QPC discussion in the case where $\Delta\theta = 0$). If we further assume the long-time limit and take the distributions of m corresponding to each qubit state to be Gaussian, we end up with the quantum limit

$$\Gamma_\varphi \geq \Gamma_{\varphi,\mathrm{info}}, \tag{3.28}$$

where the minimum dephasing rate required by the information gain of the measurement, $\Gamma_{\varphi,\mathrm{info}}$, is given by

$$\Gamma_{\varphi,\mathrm{info}} \equiv -\lim_{t\to\infty} \ln \left[\int dm \sqrt{p_\uparrow(m) p_\downarrow(m)} \right] = \frac{1}{4} \frac{\left(\langle \hat{I} \rangle_\uparrow - \langle \hat{I} \rangle_\downarrow \right)^2}{\bar{S}_{II,\uparrow} + \bar{S}_{II,\uparrow}}. \tag{3.29}$$

In this expression, we have allowed for the fact that detector output noise could be different in the two qubit states. As usual, taking the long-time limit implies that only the zero-frequency noise correlators enter this definition.

4

Quantum limit on linear amplification: the op-amp mode

4.1 Weak continuous position detection

We now turn to a more general situation, where we use our detector to amplify some time-dependent signal which is coupled to the input. For concreteness, we start with the case of continuous position detection, where the input signal is the position $\hat{x}$ of a simple harmonic oscillator of frequency Ω and mass M. The coupling Hamiltonian is thus

$$\hat{H}_{\text{int}} = A\hat{x} \cdot \hat{F}, \tag{4.1}$$

and the output $\langle \hat{I}(t) \rangle$ will respond linearly to $\langle \hat{x}(t) \rangle$. Similar to the case of qubit detection, because of the intrinsic noise in the detector output (i.e. $\bar{S}_{II}$), it will take us some time before we can resolve the signal due to the oscillator. We will focus on weak couplings, such that we only learn about the oscillator's motion on a timescale long compared to its period. As such, the goal is *not* to measure the instantaneous value of $x(t)$, but rather the slow quadrature amplitudes $X(t)$, $Y(t)$ defined via

$$\hat{x}(t) = \hat{X}(t)\cos(\Omega t) + \hat{Y}(t)\sin(\Omega t). \tag{4.2}$$

As we have already seen in great detail, the fluctuations of the input operator $\hat{F}$ correspond to a noisy backaction force which will both heat and damp the oscillator. Unlike the qubit measurement discussed in the last section, this backaction will impair our ability to measure, as the measurement is not QND: $[\hat{H}_{\text{int}}, \hat{H}_{\text{osc}}] \neq 0$. The noise in the oscillator's momentum caused by $\hat{F}$ will translate into extra position fluctuations at later times, and hence extra noise in the output of the detector. As we will see, this will place a fundamental limit on how well we can continuously monitor position. Alternatively, note that the two quadrature operators $\hat{X}$ and $\hat{Y}$ are canonically conjugate. As such, we are attempting to simultaneously measure two non-commuting observables, and hence a quantum limit is expected (i.e. we cannot know about both $\hat{X}$ and $\hat{Y}$ to arbitrary precision).

For reasons that will become clear in later sections, we will term the amplifier operation mode used here (and consequent quantum limit) the "op-amp" mode. In this mode of operation, the detector is so weakly coupled to the signal source (i.e. the oscillator) that it has almost no effect on the total oscillator damping. This is similar to an ideal voltage op-amp, where the input impedance is extremely large, and thus there is no appreciable change in the impedance of the voltage source producing the input signal. This mode of operation will be contrasted against the scattering mode of operation, where the amplifier-detector coupling is no longer weak in the above sense.

4.1.1 Defining the quantum limit

To begin, let's treat the detector output in the presence of the oscillator as a classically noisy quantity; we will also ignore the frequency-dependence of the detector response coefficient χ_{IF} to keep things simple. Letting $x(t)$ denote the signal we are trying to measure (i.e. the position of the oscillator in the absence of any corrupting backaction effect), we then have

$$I_{\mathrm{tot}}(t) = A\chi_{IF}\left[x(t) + \delta x_{\mathrm{BA}}(t)\right] + \delta I(t) \equiv A\chi_{IF}\left[x(t) + \delta x_{\mathrm{add}}(t)\right], \tag{4.3}$$

where

$$\delta x_{\mathrm{add}}(t) = \delta x_{\mathrm{BA}}(t) + \delta x_{\mathrm{imp}}(t) \equiv \delta x_{\mathrm{BA}}(t) + \frac{\delta I(t)}{A\chi_{IF}} \tag{4.4}$$

describes the total added noise of the measurement, viewed as an equivalent position fluctuation. We see that there are two distinct contributions:

- The intrinsic output fluctuations in $\hat{I}$, $\delta I(t)$, which when referred back to the oscillator gives us the imprecision noise $\delta x_{\mathrm{imp}}(t)$. Making the coupling A (or response χ_{IF}) larger reduces the magnitude of $\delta x_{\mathrm{imp}}(t)$.
- Backaction fluctuations: noise in $\hat{F}$ drives extra position fluctuations of the resonator δx_{BA}. On a classical level, we could describe these with a Langevin equation similar to Eq. (2.18), which would give

$$\delta x_{\mathrm{BA}}[\omega] = A\chi_{xx}[\omega]\delta F[\omega], \tag{4.5}$$

  where $\chi_{xx}[\omega]$ is the oscillator's force susceptibility, and is given by

$$M\chi_{xx}[\omega] = \left(\omega^2 - \Omega^2 + i\omega\gamma_0\right)^{-1}. \tag{4.6}$$

  This contribution to the added noise scales as A, and hence gets *worse* the larger one makes A.

To optimize our measurement, we would of course like to make $\delta x_{\mathrm{add}}(t)$ as small as possible. In the absence of backaction noise, we could make the added noise arbitrarily small by just increasing the coupling strength A. However, because of backaction, the best we can do is to tune A to balance the contributions form backaction and imprecision; we will be left with something non-zero.

To state the quantum limit on position detection, we first define the measured position $x_{\mathrm{meas}}(t)$ as simply the total detector output $I_{\mathrm{tot}}(t)$ referred back to the oscillator:

$$x_{\mathrm{meas}}(t) = I_{\mathrm{tot}}(t)/(A\chi_{IF}). \tag{4.7}$$

If there was *no* added noise, and further, if the oscillator was in thermal equilibrium at temperature T, the spectral density describing the fluctuations $\delta x_{\mathrm{meas}}(t)$ would simply be the equilibrium fluctuations of the oscillator, as given by the fluctuation-dissipation theorem:

$$\bar{S}_{xx}^{\text{meas}}[\omega] = \bar{S}_{xx}^{\text{eq}}[\omega, T] = \hbar \coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right)[-\text{Im }\chi_{xx}[\omega]] \tag{4.8}$$

$$= \frac{x_{\text{ZPF}}^2(1+2n_B)}{2}\sum_{\sigma=\pm}\frac{\gamma_0}{(\omega-\sigma\Omega)^2+(\gamma_0/2)^2}. \tag{4.9}$$

Here, γ_0 is the intrinsic damping rate of the oscillator, which we have assumed to be $\ll \Omega$.

Including the added noise, and for the moment ignoring the possibility of any additional oscillator damping due to the coupling to the detector, the above result becomes

$$\bar{S}_{xx}^{\text{meas}}[\omega] = \bar{S}_{xx}^{\text{eq}}[\omega, T] + \bar{S}_{xx}^{\text{add}}[\omega] \tag{4.10}$$

where the last term is the spectral density of the added noise (both backaction and imprecision noise).

We can now, finally, state the standard quantum limit on continuous position detection (which is equivalent to that on linear, phase-preserving amplification): at each frequency ω, we must have

$$\bar{S}_{xx}^{\text{add}}[\omega] \geq \bar{S}_{xx}^{\text{eq}}[\omega, T=0]. \tag{4.11}$$

The spectral density of the added noise cannot be made arbitrarily small: at each frequency, it must be at least as large as the corresponding zero-point noise.

We now refine the above result to include the presence of back-action damping of the oscillator (at a rate γ_{BA}) due to the coupling to the detector. Such damping is described by the asymmetry of the detector's $S_{FF}[\omega]$ quantum noise spectrum, as in Eq. (2.25). Including non-zero backaction damping, the added noise is defined as

$$\bar{S}_{xx}^{\text{meas}}[\omega] = \frac{\gamma_0}{\gamma_{\text{BA}}+\gamma_0}\bar{S}_{xx}^{\text{eq}}[\omega, T] + \bar{S}_{xx}^{\text{add}}[\omega], \tag{4.12}$$

where the susceptibility χ_{xx} now involves the total damping of the oscillator, i.e.:

$$M\chi_{xx}[\omega] = \left(\omega^2 - \Omega^2 + i\omega(\gamma_0+\gamma_{\text{BA}})\right)^{-1}. \tag{4.13}$$

With this definition, the quantum limit on the added noise is unchanged from the limit stated in Eq. (4.11).

4.2 A possible correlation-based loophole?

Our heuristic formulation of the quantum limit naturally leads to a possible concern. Even though quantum mechanics may require a position measurement to have a backaction (as position and momentum are conjugate quantities), couldn't this backaction noise be perfectly anti-correlated with the imprecision noise? If this were the case, the added noise $\delta x(t)$ (which is the sum of the two contributions, c.f. Eq. (4.4)) could be made to vanish.

One might hope that this sort of loophole would be explicitly forbidden by the quantum noise inequality of Eq. (2.47). However, this is not the case. Even in the ideal case of zero reverse gain, one achieve a situation where backaction and imprecision are perfectly correlated at a given non-zero frequency ω. One needs:

- The correlator $\bar{S}_{IF}[\omega]$ should purely imaginary; this implies that the part of $F(t)$ that is correlated with $I(t)$ is 90 degrees out of phase. Note that $\bar{S}_{IF}[\omega]$ can only be imaginary at non-zero frequencies.
- The magnitude of $\bar{S}_{IF}[\omega]$ should be larger than $\hbar/2$

Under these circumstances, one can verify that there is no additional quantum constrain on the noise beyond what exists classically, and hence the perfect correlation condition of $\bar{S}_{FF}[\omega]\bar{S}_{II}[\omega] = |\bar{S}_{IF}[\omega]|^2$ is allowable. The $\pi/2$ phase of the backaction-imprecision correlations are precisely what is needed to make $\delta x_{\text{add}}[\omega]$ vanish at the oscillator resonance, $\omega = \Omega$.

As might be expected, this seeming loophole is *not* a route towards amplification free from any quantum constraints. The problem is that we have not been sufficiently careful to specify what we want our detector to do, namely the condition that the detector *amplifies* the motion of the oscillator– the signal should be "bigger" at the output than it is at the input. It is only when we insist on amplification that there are quantum constraints on added noise; a passive transducer need not add any noise. On a heuristic level, one could view amplification as an effective expansion of the phase space of the oscillator. Such a pure expansion is of course forbidden by Liouville's theorem, which tells us that volume in phase space in conserved. The way out is to introduce additional degrees of freedom, such that for these degrees of freedom phase space contracts. Quantum mechanically such degrees of freedom necessarily have noise associated with them (at the very least, zero-point noise); this then is the source of the limit on added noise. We will see that heuristic argument can be converted into a rigorous formulation of the quantum limit (albeit of a different sort) in Sec. 5.

More concretely, we ned to define what we mean by amplification in our linear response detector. We can then rigorously insist that our detector amplifies. The result will be additional constraints beyond the quantum noise inequality of Eq. (2.47) which make the perfect correlation described above impossible.

4.3 Power gain

To be able to say that our detector truly amplifies the motion of the oscillator, it is not sufficient to simply say the response function χ_{IF} must be large (note that χ_{IF} is not dimensionless!). Instead, true amplification requires that the *power* delivered by the detector to a following amplifier be much larger than the power drawn by the detector at its input– i.e., the detector must have a dimensionless power gain $G_P[\omega]$ much larger than one. If the power gain was not large, we would need to worry about the next stage in the amplification of our signal, and how much noise is added in that process. Having a large power gain means that by the time our signal reaches

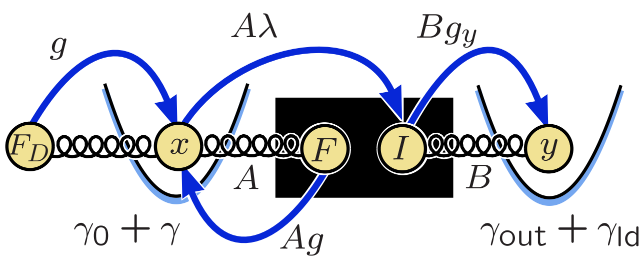

Fig. 4.1 (Color online) Schematic of a generic linear-response position detector, where an auxiliary oscillator y is driven by the detector output.

the following amplifier, it is so large that the added noise of this following amplifier is unimportant

To make the above more precise, we start with the ideal case of no reverse gain, $\chi_{FI} = 0$. We will define the power gain $G_P[\omega]$ of our generic position detector in a way that is analogous to the power gain of a voltage amplifier. Imagine we drive the oscillator we are trying to measure (whose position is x) with a force $2F_D \cos\omega t$; this will cause the output of our detector $\langle \hat{I}(t)\rangle$ to also oscillate at frequency ω. To optimally detect this signal in the detector output, we further couple the detector output I to a second oscillator with natural frequency ω, mass M, and position y: there is a new coupling term in our Hamiltonian, $H'_{int} = B\hat{I}\cdot\hat{y}$, where B is a coupling strength. The oscillations in $\langle I(t)\rangle$ will now act as a driving force on the auxiliary oscillator y (see Fig 4.1). We can consider the auxiliary oscillator y as a "load" we are trying to drive with the output of our detector.

To find the power gain, we need to consider both P_{out}, the power supplied to the output oscillator y from the detector, and P_{in}, the power fed into the input of the amplifier. Consider first P_{in}. This is simply the time-averaged power dissipation of the input oscillator x caused by the back-action damping $\gamma_{\mathrm{BA}}[\omega]$. Using a bar to denote a time average, we have

$$P_{\mathrm{in}} \equiv M\gamma_{\mathrm{BA}}[\omega]\cdot\overline{\dot{x}^2} = M\gamma_{\mathrm{BA}}[\omega]\omega^2|\chi_{xx}[\omega]|^2 F_D^2. \tag{4.14}$$

Note that the oscillator susceptibility $\chi_{xx}[\omega]$ includes the effects of γ_{BA}, c.f. Eq. (4.13).

Next, we need to consider the power supplied to the "load" oscillator y at the detector output. This oscillator will have some intrinsic, detector-independent damping γ_{ld}, as well as a back-action damping γ_{out}. In the same way that the back-action damping γ_{BA} of the input oscillator x is determined by the quantum noise in $\hat{F}$ (cf. Eq. (2.25)), the back-action damping of the load oscillator y is determined by the quantum noise in the output operator $\hat{I}$:

$$\begin{aligned}\gamma_{\mathrm{out}}[\omega] &= \frac{B^2}{M\omega}\left[-\mathrm{Im}\ \chi_{II}[\omega]\right]\\ &= \frac{B^2}{M\hbar\omega}\left[\frac{S_{II}[\omega]-S_{II}[-\omega]}{2}\right],\end{aligned} \tag{4.15}$$

where χ_{II} is the linear-response susceptibility which determines how $\langle \hat{I} \rangle$ responds to a perturbation coupling to $\hat{I}$:

$$\chi_{II}[\omega] = -\frac{i}{\hbar}\int_0^\infty dt \left\langle \left[\hat{I}(t), \hat{I}(0)\right] \right\rangle e^{i\omega t}. \tag{4.16}$$

As the oscillator y is being driven on resonance, the relation between y and I is given by $y[\omega] = \chi_{yy}[\omega]I[\omega]$ with $\chi_{yy}[\omega] = -i[\omega M\gamma_{\rm out}[\omega]]^{-1}$. From conservation of energy, we have that the *net* power flow into the output oscillator from the detector is equal to the power dissipated out of the oscillator through the intrinsic damping $\gamma_{\rm ld}$. We thus have

$$\begin{aligned} P_{\rm out} &\equiv M\gamma_{\rm ld}\cdot \overline{\dot{y}^2} \\ &= M\gamma_{\rm ld}\omega^2|\chi_{yy}[\omega]|^2 \cdot |BA\chi_{IF}\chi_{xx}[\omega]F_D|^2 \\ &= \frac{1}{M}\frac{\gamma_{\rm ld}}{(\gamma_{\rm ld}+\gamma_{\rm out}[\omega])^2}\cdot |BA\chi_{IF}\chi_{xx}[\omega]F_D|^2. \end{aligned} \tag{4.17}$$

Using the above definitions, we find that the ratio between $P_{\rm out}$ and $P_{\rm in}$ is independent of γ_0, but depends on $\gamma_{\rm ld}$:

$$\frac{P_{\rm out}}{P_{\rm in}} = \frac{1}{M^2\omega^2}\frac{A^2B^2|\chi_{IF}[\omega]|^2}{\gamma_{\rm out}[\omega]\gamma_{\rm BA}[\omega]}\frac{\gamma_{\rm ld}/\gamma_{\rm out}[\omega]}{(1+\gamma_{\rm ld}/\gamma_{\rm out}[\omega])^2}. \tag{4.18}$$

We now define the detector power gain $G_P[\omega]$ as the value of this ratio maximized over the choice of $\gamma_{\rm ld}$. The maximum occurs for $\gamma_{\rm ld} = \gamma_{\rm out}[\omega]$ (i.e. the load oscillator is "matched" to the output of the detector), resulting in:

$$\begin{aligned} G_P[\omega] &\equiv \max\left[\frac{P_{\rm out}}{P_{\rm in}}\right] \\ &= \frac{1}{4M^2\omega^2}\frac{A^2B^2|\chi_{IF}|^2}{\gamma_{\rm out}\gamma_{\rm BA}} \\ &= \frac{|\chi_{IF}[\omega]|^2}{4{\rm Im}\ \chi_{FF}[\omega]\cdot {\rm Im}\ \chi_{II}[\omega]} \end{aligned} \tag{4.19}$$

In the last line, we have used the relation between the damping rates $\gamma_{\rm BA}[\omega]$ and $\gamma_{\rm out}[\omega]$ and the linear-response susceptibilities $\chi_{FF}[\omega]$ and $\chi_{II}[\omega]$, c.f. Eq. (2.29). We thus find that the power gain is a simple dimensionless ratio formed by the three different response coefficients characterizing the detector, and is independent of the coupling constants A and B. As we will see, it is completely analogous to the power gain of a voltage amplifier, which is also determined by three parameters: the voltage gain, the input impedance and the output impedance.

Finally, we note that the above results can be generalized to include a non-zero detector reverse gain, χ_{FI}, see (Clerk *et al.*, 2010). We saw previously in Sec. 4.2 that if $\chi_{FI} = \chi_{IF}^*$, then there is no additional quantum constraint on the noise beyond what exists classically. In this case of a perfectly symmetric detector, one can show that the power gain is at most equal to one: true amplification is never possible in this case.

4.4 Simplifications for a detector with ideal quantum noise and large power gain

Requiring both the quantum noise inequality in Eq. (2.47) to be saturated at frequency ω as well as a large power gain (i.e. $G_P[\omega] \gg 1$) leads to some important additional constraints on the detector, as derived in Appendix I of (Clerk *et al.*, 2010):

- $(2/\hbar)\mathrm{Im}\,\bar{S}_{zF}[\omega]$ is small like $1/\sqrt{G_P[\omega]}$. Hence, the possibility of having a perfect backaction-imprecision noise correlations as discussed in Sec. 4.2 is excluded.
- The detector's effective temperature must be much larger than $\hbar\omega$; one finds

$$k_{\mathrm{B}} T_{\mathrm{eff}}[\omega] \sim \sqrt{G_P[\omega]}\hbar\omega. \tag{4.20}$$

  Conversely, it is the largeness of the detector's effective temperature that allows it to have a large power gain.

4.5 Derivation of the quantum limit

We now turn to a rigorous proof of the quantum limit on the added noise given in Eq. (4.11). From the classical-looking Eq. (4.4), we expect that the symmetrized quantum noise spectral density describing the added noise will be given by

$$\bar{S}_{xx,\mathrm{add}}[\omega] = \frac{\bar{S}_{II}}{|\chi_{IF}|^2 A^2} + A^2 \left|\chi_{xx}\right|^2 \bar{S}_{FF} + \frac{2\mathrm{Re}\ \left[\chi_{IF}^* \left(\chi_{xx}\right)^* \bar{S}_{IF}\right]}{|\chi_{IF}|^2} \tag{4.21}$$

$$= \frac{\bar{S}_{zz}}{A^2} + A^2 \left|\chi_{xx}\right|^2 \bar{S}_{FF} + 2\mathrm{Re}\ \left[\left(\chi_{xx}\right)^* \bar{S}_{zF}\right]. \tag{4.22}$$

In the second line, we have introduced the imprecision noise $\bar{S}_{zz}$ and imprecision backaction correlation $\bar{S}_{zF}$ as in Eq. (2.41). We have also omitted writing the explicit frequency dependence of the gain χ_{IF}, susceptibility χ_{xx}, and noise correlators; they should all be evaluated at the frequency ω. Finally, the oscillator susceptibility χ_{xx} here is given by Eq. (4.13), and includes the effects of backaction damping. While we have motivated this equation from a seemingly classical noise description, the full quantum theory also yields the same result: one simply calculates the detector output noise perturbatively in the coupling to the oscillator (Clerk, 2004).

The first step in determining the limit on the added noise is to consider its dependence on the coupling strength strength A. If we ignore for a moment the detector-dependent damping of the oscillator, there will be an optimal value of the coupling strength A which corresponds to a trade-off between imprecision noise and back-action (i.e. first and second terms in Eq. (4.21)). We would thus expect $\bar{S}_{xx,\mathrm{add}}[\omega]$ to attain a minimum value at an optimal choice of coupling $A = A_{\mathrm{opt}}$ where both these terms make equal contributions. Defining $\phi[\omega] = \arg \chi_{xx}[\omega]$, we thus have the bound

$$\bar{S}_{xx,\mathrm{add}}[\omega] \geq 2|\chi_{xx}[\omega]| \left(\sqrt{\bar{S}_{zz}\bar{S}_{FF}} + \mathrm{Re}\ \left[e^{-i\phi[\omega]}\bar{S}_{zF}\right]\right), \tag{4.23}$$

where the minimum value at frequency ω is achieved when

$$A_{\mathrm{opt}}^2 = \sqrt{\frac{\bar{S}_{zz}[\omega]}{|\chi_{xx}[\omega]|^2 \bar{S}_{FF}[\omega]}}. \tag{4.24}$$

Using the inequality $X^2 + Y^2 \geq 2|XY|$ we see that this value serves as a lower bound on $\bar{S}_{xx,\mathrm{add}}$ even in the presence of detector-dependent damping. In the case where the detector-dependent damping is negligible, the RHS of Eq. (4.23) is independent of A, and thus Eq. (4.24) can be satisfied by simply tuning the coupling strength A; in the more general case where there is detector-dependent damping, the RHS is also a function of A (through the response function $\chi_{xx}[\omega]$), and it may no longer be possible to achieve Eq. (4.24) by simply tuning A.

While Eq. (4.23) is certainly a bound on the added displacement noise $\bar{S}_{xx,\mathrm{add}}[\omega]$, it does not in itself represent the quantum limit. Reaching the quantum limit requires more than simply balancing the detector back-action and intrinsic output noises (i.e. the first two terms in Eq. (4.21)); *one also needs a detector with "quantum-ideal" noise properties, that is a detector which optimizes Eq. (2.47)*. Using the quantum noise constraint of Eq. (2.47) to further bound $\bar{S}_{xx,\mathrm{add}}[\omega]$, we obtain

$$\bar{S}_{xx,\mathrm{add}}[\omega] \geq \hbar \left|\chi_{xx}[\omega]\right| \left[\sqrt{\left(1 + \Delta\left[\frac{\bar{S}_{zF}}{\hbar/2}\right]\right) + \left|\frac{\bar{S}_{zF}}{\hbar/2}\right|^2} + \frac{\mathrm{Re}\ \left[e^{-i\phi[\omega]}\bar{S}_{zF}\right]}{\hbar/2}\right], \quad (4.25)$$

where the function $\Delta[z]$ is defined in Eq. (2.48). The minimum value of $\bar{S}_{xx,\mathrm{add}}[\omega]$ in Eq. (4.25) is now achieved when one has *both* an optimal coupling (i.e. Eq. (4.24)) *and* a quantum limited detector, that is one which satisfies Eq. (2.47) as an equality.

Next, we consider the relevant case where our detector is a good amplifier and has a power gain $G_P[\omega] \gg 1$ over the width of the oscillator resonance. As we have discussed, this implies that the ratio $\bar{S}_{zF}$ is purely real, up to small $1/G_P$ corrections (see Appendix I of Ref. (Clerk *et al.*, 2010) for more details). This in turn implies that $\Delta[2\bar{S}_{zF}/\hbar] = 0$; we thus have

$$\bar{S}_{xx,\mathrm{add}}[\omega] \geq \hbar|\chi_{xx}[\omega]| \left[\sqrt{1 + \left(\frac{\bar{S}_{zF}}{\hbar/2}\right)^2} + \cos\left(\phi[\omega]\right)\frac{\bar{S}_{zF}}{\hbar/2}\right]. \quad (4.26)$$

Finally, as there is no further constraint on $\bar{S}_{zF}$ (beyond the fact that it is real), we can minimize the expression over its value. The minimum $\bar{S}_{xx,\mathrm{add}}[\omega]$ is achieved for a detector whose cross-correlator satisfies

$$\bar{S}_{zF}[\omega]\Big|_{\mathrm{optimal}} = -\frac{\hbar}{2}\cot\phi[\omega], \quad (4.27)$$

with the minimum value of the added noise being given precisely by

$$\bar{S}_{xx,\mathrm{add}}[\omega]\Big|_{min} = \hbar|\mathrm{Im}\ \chi_{xx}[\omega]| = \lim_{T\to 0}\bar{S}_{xx,\mathrm{eq}}[\omega, T], \quad (4.28)$$

in agreement with Eq. (4.11). Thus, in the limit of a large power gain, we have that *at each frequency, the minimum displacement noise added by the detector is precisely equal to the noise arising from a zero temperature bath.* This conclusion is irrespective of the strength of the intrinsic (detector-independent) oscillator damping.

We have thus derived the amplifier quantum limit (in the context of position detection) for a two-port amplifier used in the "op-amp" mode of operation. Moreover, our derivation shows that to reach the quantum-limit on the added displacement noise $\bar{S}_{xx,\mathrm{add}}[\omega]$ with a large power gain, one needs:

1. A detector with quantum limited noise properties, that is one which optimizes the inequality of Eq. (2.47). Similar to our discussion of QND qubit detection, optimizing this inequality corresponds to the heuristic requirement of "no wasted information".
2. A coupling A which satisfies Eq. (4.24).
3. A detector cross-correlator $\bar{S}_{IF}$ which satisfies Eq. (4.27).

It is worth stressing that Eq. (4.27) implies that it will *not* in general be possible to achieve the quantum limit simultaneously at all frequencies. On resonance, this condition tells us that $\bar{S}_{zF}[\omega]$ should be zero. In contrast, far from resonance, it implies that one needs strong correlations, $\bar{S}_{zF} \gg \hbar/2$. There are systems in which one is indeed interested in minimizing the added noise far from resonance. For example, in interferometers used for gravitational wave detection, the test masses used are almost in the free-mass limit, and thus one is interested in frequencies much much larger than resonance frequency of the test mass. A way to achieve such large imprecision-backaction correlations using a nonlinear cavity detector was discussed recently in (Laflamme and Clerk, 2011).

If one focuses on optimizing $\bar{S}_{xx,\mathrm{add}}[\omega]$ at resonance (i.e. $\omega = \Omega$), and if one is using a quantum-limited detector with a large power gain ($k_\mathrm{B}T_\mathrm{eff} \gg \hbar\Omega$), the remaining condition on the coupling A, Eq. (4.24), may be written

$$\frac{\gamma_\mathrm{BA}[A_\mathrm{opt}]}{\gamma_0 + \gamma_\mathrm{BA}[A_\mathrm{opt}]} = \frac{\hbar\Omega}{4k_\mathrm{B}T_\mathrm{eff}}. \tag{4.29}$$

As $\gamma_\mathrm{BA}[A] \propto A^2$ is the detector-dependent damping of the oscillator, we thus have that *to achieve the quantum-limited value of* $\bar{S}_{xx,\mathrm{add}}[\Omega]$ *with a large power gain, one needs the intrinsic damping of the oscillator to be much larger than the detector-dependent damping.* The detector-dependent damping must be small enough to compensate the large effective temperature of the detector; if the bath temperature satisfies $\hbar\Omega/k_\mathrm{B} \ll T_\mathrm{bath} \ll T_\mathrm{eff}$, Eq. (4.29) implies that at the quantum limit, the temperature of the oscillator will be given by

$$T_\mathrm{osc} \equiv \frac{\gamma_\mathrm{BA} \cdot T_\mathrm{eff} + \gamma_0 \cdot T_\mathrm{bath}}{\gamma_\mathrm{BA} + \gamma_0} \to \frac{\hbar\Omega}{4k_\mathrm{B}} + T_\mathrm{bath}. \tag{4.30}$$

Thus, at the quantum limit and for large T_eff, the detector raises the oscillator's temperature by $\hbar\Omega/4k_\mathrm{B}$.[1] As expected, this additional heating is only *half* the zero-point energy; in contrast, the quantum-limited value of $\bar{S}_{xx,\mathrm{add}}[\omega]$ corresponds to the full zero-point result, as it also includes the contribution of the intrinsic output noise of the detector.

Finally, we return to Eq. (4.25); this is the constraint on the added noise $\bar{S}_{xx,\mathrm{add}}[\omega]$ *before* we assumed our detector to have a large power gain, and consequently a large

[1] If in contrast our oscillator was initially at zero temperature (i.e. $T_\mathrm{bath} = 0$), one finds that the effect of the back-action (at the quantum limit and for $G_P \gg 1$) is to heat the oscillator to a temperature $\hbar\Omega/(k_\mathrm{B}\ln 5) \simeq 0.62\hbar\Omega/k_\mathrm{B}$.

$T_{\rm eff}$. Note crucially that *if* we did not require a large power gain, then there need not be *any* added noise. Without the assumption of a large power gain, the ratio $\bar{S}_{IF}/\chi_{IF}$ can be made imaginary with a large magnitude. In this limit, $1 + \Delta[2\bar{S}_{IF}/\chi_{IF}] \to 0$: the quantum constraint on the amplifier noises (e.g. the RHS of Eq. (2.47)) vanishes. One can then easily use Eq. (4.25) to show that the added noise $\bar{S}_{xx,\rm add}[\omega]$ can be zero.

4.6 Noise temperature

There is an alternate but roughly equivalent way of phrasing the amplifier quantum limit we have been discussing, where the crucial quantity to be bounded is the so-called "noise temperature" $T_N[\omega]$. For a given frequency ω, this is defined via the equation

$$\bar{S}^{\rm eq}_{xx}[\omega, T + T_N[\omega]] \equiv \bar{S}^{\rm eq}_{xx}[\omega, T] + \bar{S}^{\rm add}_{xx}[\omega], \tag{4.31}$$

in the limit where $T \gg \hbar\omega$ (which ensures that the definition of T_N is independent of T) [2]. In words, the added noise position fluctuations at a frequency ω can be viewed as an effective heating of the oscillator from a temperature T to a temperature $T+T_N[\omega]$. One finds

$$\frac{2k_{\rm B}T_N[\omega]}{\omega} = \frac{\bar{S}^{\rm add}_{xx}[\omega]}{-\mathrm{Im}\ \chi_{xx}[\omega]} = \frac{1}{-\mathrm{Im}\ \chi_{xx}[\omega]}\left(\frac{S_{zz}}{A^2} + A^2\left|\chi_{xx}\right|^2\bar{S}_{FF} + 2\mathrm{Re}\ \left(\chi^*_{xx}\bar{S}_{zF}\right)\right). \tag{4.32}$$

Our bound on the added noise spectral density (in the high power gain limit) immediately implies that for large power gain,

$$k_{\rm B}T_N[\omega] \geq \hbar\omega/2. \tag{4.33}$$

In complete analogy to the quantum limit on the added noise, reaching the quantum limit on the noise temperature first requires one to balance the contributions from imprecision and backaction by, e.g., tuning the value of the coupling A. For the noise temperature, note that the imprecision contribution scales as $1/|\chi_{xx}|$, whereas the backaction contribution scales as $|\chi_{xx}|$. Thus, for the noise temperature, one could keep A fixed, and balance backaction and imprecision noise contributions by tuning the value of the oscillator susceptibility $|\chi_{xx}[\omega]|$ (e.g. imagine one could tune the value of Ω). In a similar fashion, as opposed to tuning the value of the detector cross-correlator $\bar{S}_{zF}$, one could tune the phase of χ_{xx}. While this way of thinking about the optimization may seem quite unnatural in the context of a position detector, it is completely natural if we now think of our detector as a voltage amplifier. As we will see, in this case the role of χ_{xx} is played $Z_{\rm src}$, the source impedance of the system producing the signal that is to be amplified.

4.7 Quantum limit on an "op-amp" style voltage amplifier

We now use our general linear-response machinery to tackle the quantum limit on the noise temperature of a voltage amplifier, again used in the "op-amp" mode of

[2] Other, less common, definitions of the noise temperature also exist in the literature. For example, (Caves, 1982) defines the noise temperature using Eq. (4.31)), but takes the initial temperature T to be zero. With this definition, the quantum limit takes the somewhat awkward form $k_B T_N[\omega] \geq \hbar\omega/\ln(3)$.

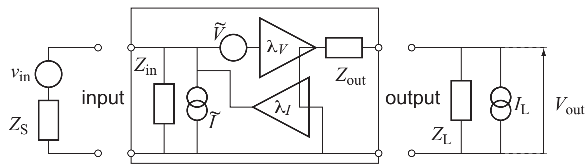

Fig. 4.2 Schematic of a linear voltage amplifier, including a reverse gain λ_I. $\tilde{V}$ and $\tilde{I}$ represent the standard voltage and current noises of the amplifier, as discussed further in the text. The case with reverse gain is discussed in Sec. 5.2.

operation. For the most part, this just involves a simple relabelling of various quantities in our position detector; this is done in Table 4.1. The input signal to the detector is now a voltage $v_{\rm in}(t)$, and the output quantity that is read out is also a voltage: the output operator $\hat{I}$ thus now becomes an operator $\hat{V}_{\rm out}$.

Fig. 4.2 shows a standard schematic description of a voltage amplifier. The input voltage to be amplified $v_{\rm in}(t)$ is produced by a circuit which has a Thevenin-equivalent impedance $Z_{\rm s}$, the source impedance. The amplifier itself has an input impedance $Z_{\rm in}$ and an output impedance $Z_{\rm out}$, as well as a voltage gain coefficient λ_V: assuming no current is drawn at the output (i.e. $Z_{\rm load} \to \infty$ in Fig. 4.2), the output voltage $V_{\rm out}(t)$ is simply λ_V times the voltage across the input terminals of the amplifier. We focus here on the simple case of no reverse gain.

The added noise of the amplifier is standardly represented by two noise sources placed at the amplifier input. There is both a voltage noise source $\tilde{V}(t)$ in series with the input voltage source, and a current noise source $\tilde{I}(t)$ in parallel with input voltage source (Fig. 4.2). The voltage noise produces a fluctuating voltage $\tilde{V}(t)$ (spectral density $\bar{S}_{\tilde{V}\tilde{V}}[\omega]$) which simply adds to the signal voltage at the amplifier input, and is amplified at the output; as such, it is completely analogous to the imprecision noise $\bar{S}_{zz} = \bar{S}_{II}/|\chi_{IF}|^2$ of our linear response detector. In contrast, the current noise source of the voltage amplifier represents back-action: this fluctuating current (spectral density $\bar{S}_{\tilde{I}\tilde{I}}[\omega]$) flows back across the parallel combination of the source impedance and amplifier input impedance, producing an additional fluctuating voltage at its input. The current noise is thus analogous to the back-action noise $\bar{S}_{FF}$ of our generic linear response detector.

Putting the above together, and treating for the moment the output voltage as a classically noisy variable, we have

$$\begin{aligned} v_{\rm out,tot}(t) &= \lambda_V \left[\frac{Z_{\rm in}}{Z_{\rm in} + Z_{\rm s}} \left[v_{\rm in}(t) + \tilde{V}(t) \right] - \frac{Z_{\rm s} Z_{\rm in}}{Z_{\rm s} + Z_{\rm in}} \tilde{I}(t) \right] \\ &\simeq \lambda_V \left[v_{\rm in}(t) + \tilde{V}(t) - Z_{\rm s} \tilde{I}(t) \right]. \end{aligned} \tag{4.34}$$

In the second line, we have taken the usual limit of an ideal voltage amplifier which has an infinite input impedance (i.e. the amplifier draws zero current). We are left with an equation that is completely analogous to the corresponding Eq. (4.3) for a position detector. For simplicity, we have ignored any frequency dependence of $\lambda_V, Z_{\rm s}$ and $Z_{\rm in}$; these are easily restored.

Table 4.1 Correspondence between position detector and voltage amplified.

Quantity	Position detector	Voltage amplifier
Input signal	$\hat{x}(t)$	$\hat{v}_{\mathrm{in}}(t)$
Output operator	$\hat{I}(t)$	$\hat{V}_{\mathrm{out}}(t)$
Backaction operator (i.e. quantity which couples to input signal)	$\hat{F}[\omega]$	$\hat{Q}[\omega] = \hat{\tilde{I}}[\omega]/(-i\omega)$
Linear gain coefficient	$\chi_{IF}[\omega]$	$\lambda_V[\omega] = \chi_{V_{\mathrm{out}}Q}[\omega]$
Imprecision noise operator	$z[\omega] = I[\omega]/\chi_{IF}[\omega]$	$\tilde{V}[\omega] = \hat{V}_{\mathrm{out}}[\omega]/\lambda_V[\omega]$
Imprecision noise	$\bar{S}_{zz}[\omega] = \bar{S}_{II}[\omega]/\lvert\chi_{IF}[\omega]\rvert^2$	$\bar{S}_{\tilde{V}\tilde{V}}[\omega]$
Backaction noise	$\bar{S}_{FF}[\omega]$	$\bar{S}_{\tilde{I}\tilde{I}}[\omega]/\omega^2$
Backaction-imprecision correlator	$\bar{S}_{zF}[\omega]$	$\bar{S}_{\tilde{V}\tilde{I}}[\omega]/(i\omega)$
Detector-induced input dissipation	$-\mathrm{Im}\ \chi_{FF}[\omega] = M\omega\gamma_{\mathrm{BA}}[\omega]$	$-\mathrm{Im}\ \chi_{QQ}[\omega] = \frac{1}{\omega}\mathrm{Re}\frac{1}{Z_{\mathrm{in}}[\omega]}$
Detector-induced output dissipation	$-\mathrm{Im}\ \chi_{II}[\omega] = M\omega\gamma_y[\omega]$	$-\mathrm{Im}\ \chi_{V_{\mathrm{out}}V_{\mathrm{out}}}[\omega] = \omega\mathrm{Re}Z_{\mathrm{out}}[\omega]$
Power gain G_P	$\frac{\lvert\chi_{IF}[\omega]\rvert^2/4}{\mathrm{Im}\ \chi_{FF}[\omega]\cdot\mathrm{Im}\ \chi_{II}[\omega]}$	$\frac{\lvert\lambda_V\rvert^2/4}{\mathrm{Re}\ (Z_{\mathrm{out}})\mathrm{Re}\ (1/Z_{\mathrm{in}})}$

If we refer the output voltage fluctuations back to the input (via λ_V) to obtain the measured voltage fluctuations, we find they are described by a spectral density

$$\bar{S}_{VV,\mathrm{tot}}[\omega] = \bar{S}_{v_{\mathrm{in}}v_{\mathrm{in}}}[\omega] + \bar{S}_{VV,\mathrm{add}}[\omega]. \tag{4.35}$$

As in our discussion of position detection, we have used the fact that quantum expressions for symmetrized noise densities will match what is expected from the classical theory. Here $\bar{S}_{v_{\mathrm{in}}v_{\mathrm{in}}}[\omega]$ is the spectral density of the voltage fluctuations of the input signal $v_{\mathrm{in}}(t)$; assuming the signal source is in equilibrium at temperature T, they will be given by

$$\bar{S}_{v_{\mathrm{in}}v_{\mathrm{in}}}[\omega] = \mathrm{Re}\ Z_s[\omega]\hbar\omega\coth\left(\hbar\omega/2k_{\mathrm{B}}T\right), \tag{4.36}$$

consistent with the fluctuation dissipation theorem. In contrast, $\bar{S}_{VV,\mathrm{add}}$ is the total amplifier added noise (referred to the input), and has both a backaction and imprecision noise contribution:

$$\bar{S}_{VV,\mathrm{add}}[\omega] = \bar{S}_{\tilde{V}\tilde{V}} + \left|Z_{\mathrm{s}}\right|^2 \bar{S}_{\tilde{I}\tilde{I}} - 2\mathrm{Re}\left[Z_{\mathrm{s}}^* \bar{S}_{\tilde{V}\tilde{I}}\right]. \tag{4.37}$$

For clarity, we have dropped the frequency index for the spectral densities appearing on the RHS of this equation.

Finally, we can define the noise temperature exactly as in Eq.(4.32) by viewing the added noise at frequency ω as being due to an effective heating of the source from temperature $T \gg \hbar\omega/k_{\mathrm{B}}$ to $T + T_N[\omega]$. Writing $Z_{\mathrm{s}} = |Z_{\mathrm{s}}|e^{i\phi}$, we find:

$$2k_{\mathrm{B}}T_{\mathrm{N}} = \frac{1}{\cos\phi}\left[\frac{\bar{S}_{\tilde{V}\tilde{V}}}{|Z_{\mathrm{s}}|} + |Z_{\mathrm{s}}|\bar{S}_{\tilde{I}\tilde{I}} - 2\mathrm{Re}\ \left(e^{-i\phi}\bar{S}_{\tilde{V}\tilde{I}}\right)\right] \tag{4.38}$$

We can now apply the same line of reasoning as we used for the T_N of a position detector, and thus in the large power-gain limit, the quantum limit of Eq. (4.33) applies to $T_N[\omega]$. Achieving the quantum limit requires optimizing the quantum noise constraint of Eq. (2.47), which here takes the form

$$\bar{S}_{\tilde{V}\tilde{V}}[\omega]\bar{S}_{\tilde{I}\tilde{I}}[\omega] - \left[\mathrm{Im}\ \bar{S}_{\tilde{V}\tilde{I}}\right]^2 \geq \left(\frac{\hbar\omega}{2}\right)^2. \tag{4.39}$$

In addition, one also needs to balance the contribution of backaction and imprecision (c.f. Eq. (4.23)), as well as properly take advantage of any backaction-imprecision correlations (c.f. Eq. (4.27)). Unlike the oscillator case, this is most naturally done by tuning both the magnitude and phase of the source impedance Z_s. If one simply minimizes $T_N[\omega]$ with respect to Z_s, one finds a completely classical minimum bound on T_N,

$$k_\mathrm{B}T_\mathrm{N} \geq \sqrt{\bar{S}_{\tilde{V}\tilde{V}}\bar{S}_{\tilde{I}\tilde{I}} - \left[\mathrm{Im}\ \bar{S}_{\tilde{V}\tilde{I}}\right]^2} - \mathrm{Re}\ \bar{S}_{\tilde{V}\tilde{I}}, \tag{4.40}$$

where the minimum is achieved for an optimal source impedance which satisfies:

$$|Z_\mathrm{s}[\omega]|_\mathrm{opt} = \sqrt{\frac{\bar{S}_{\tilde{V}\tilde{V}}[\omega]}{\bar{S}_{\tilde{I}\tilde{I}}[\omega]}} \equiv Z_\mathrm{N} \tag{4.41}$$

$$\sin\phi[\omega]\big|_\mathrm{opt} = -\frac{\mathrm{Im}\ \bar{S}_{\tilde{V}\tilde{I}}[\omega]}{\sqrt{\bar{S}_{\tilde{V}\tilde{V}}[\omega]\bar{S}_{\tilde{I}\tilde{I}}[\omega]}} \tag{4.42}$$

Recall that in the position detector, achieving the quantum limit required a coupling A so weak that the detector-induced damping γ_BA was much less than the intrinsic oscillator damping (cf. Eq. (4.29)). Similarly, in the present case, one can show (see (Clerk *et al.*, 2010)) that for a large power-gain amplifier with ideal quantum noise, the noise impedance Z_N satisfies

$$\left|\frac{Z_\mathrm{N}[\omega]}{\mathrm{Re}\ Z_\mathrm{in}[\omega]}\right| = \frac{1}{2\sqrt{G_P[\omega]}} \ll 1. \tag{4.43}$$

It follows that $|Z_\mathrm{N}| \ll |Z_\mathrm{in}|$ in the large power gain, large effective temperature regime of interest, thus justifying the form of Eq. (4.37).

4.7.1 Role of noise cross-correlations

Before leaving the topic of a linear voltage amplifier, we pause to note the role of cross-correlations in current and voltage noise in reaching the quantum limit. First, note from Eq. (4.42) that in both the classical and quantum treatments, the noise impedance Z_N of the amplifier will have a reactive part (i.e. Im $Z_\mathrm{N} \neq 0$) if there are out-of-phase correlations between the amplifier's current and voltage noises (i.e. if Im $S_{VI} \neq 0$). Thus, if such correlations exist, it will not be possible to minimize the noise temperature (and hence, reach the quantum limit), if one uses a purely real source impedance Z_s.

More significantly, note that the final classical expression for the noise temperature T_N explicitly involves the real part of the S_{VI} correlator (cf. Eq. (4.40)). In contrast, we have argued that in the quantum case, Re $\bar{S}_{VI}$ *must* be zero if one wishes to reach the quantum limit while having a large power gain (cf. Sec. 4.4 and Appendix I of (Clerk *et al.*, 2010)); as such, this quantity does not appear in the final expression for the minimal T_{N}. It also follows that *to reach the quantum limit while having a large power gain, an amplifier cannot have significant in-phase correlations between its current and voltage noise.*

This last statement can be given a heuristic explanation. If there are out-of-phase correlations between current and voltage noise, we can easily make use of these by appropriately choosing our source impedance. However, if there are in-phase correlations between current and voltage noise, we cannot use these simply by tuning the source impedance. We *could* however have used them by implementing feedback in our amplifier. The fact that we have not done this means that these correlations represent a kind of missing information; as a result, we must necessarily miss the quantum limit. In Sec. 5.2.2, we explicitly give an example of a voltage amplifier which misses the quantum limit due to the presence of in-phase current and voltage fluctuations.

5

Quantum limit on a linear-amplifier: scattering mode

We now consider an alternate mode of amplifier operation, where the input signal is the amplitude of a wave incident on the amplifier, and the output signal is the amplitude of a wave leaving the amplifier. Unlike the op-amp mode considered above, backaction is irrelevant here: the assumption is that the input signal entering the amplifier is completely insensitive to any fluctuations emanating from the amplifier. To achieve this, one typically has to impedance match the signal source to the amplifier input so that there are no reflections. This is very different from the op-amp mode, where the amplifier input impedance is much larger than the source impedance. This condition ensures that in the op-amp mode, the coupling to the amplifier only weakly increases the dissipation of the signal source. In contrast, the impedance matching implies that the coupling to the amplifier will have a more pronounced impact on the signal source dissipation.

While the quantum limit on the added noise of an amplifier in this scattering mode of operation has the same form as that in the op-amp mode, we will see that they are not equivalent: an amplifier can be quantum limited in one mode, but fail to reach the quantum limit in the other mode.

5.1 Caves-Haus formulation of the scattering-mode quantum limit

The derivation of the amplifier quantum limit in the scattering mode of operation is in many ways better known that the op-amp quantum limit presented above, and is simpler to present. This derivation is originally due to (Haus and Mullen, 1962), and was both clarified and extended by (Caves, 1982); the amplifier quantum limit was also motivated in a slightly different manner by (Heffner, 1962).[1]

The starting assumption of this derivation is that both the input and output ports of the amplifier can be described by sets of bosonic modes. If we focus on a narrow bandwidth centered on frequency ω, we can describe a classical signal $E(t)$ in terms of a complex number a defining the amplitude and phase of the signal (or equivalently the two quadrature amplitudes) (Haus and Mullen, 1962; Haus, 2000)

$$E(t) \propto i[ae^{-i\omega t} - a^* e^{+i\omega t}]. \tag{5.1}$$

[1]Note that (Caves, 1982) provides a thorough discussion of why the derivation of the amplifier quantum limit given in (Heffner, 1962) is not rigorously correct

In the quantum case, the two signal quadratures of $E(t)$ (i.e. the real and imaginary parts of $a(t)$) cannot be measured simultaneously because they are canonically conjugate; this is in complete analogy to a harmonic oscillator (cf. Eq. (4.2)). As a result a, a^* must be elevated to the status of photon ladder operators: $a \to \hat{a}, a^* \to \hat{a}^\dagger$.

Consider the simplest case, where there is only a single mode at both the input and output, with corresponding operators $\hat{a}$ and $\hat{b}$. It follows that the input signal into the amplifier is described by the expectation value $\langle \hat{a} \rangle$, while the output signal is described by $\langle \hat{b} \rangle$. Correspondingly, the symmetrized noise in both these quantities is described by

$$(\Delta a)^2 \equiv \frac{1}{2} \left\langle \{\hat{a}, \hat{a}^\dagger\} \right\rangle - |\langle \hat{a} \rangle|^2 , \tag{5.2}$$

with an analogous definition for $(\Delta b)^2$.

To derive a quantum limit on the added noise of the amplifier, one uses two simple facts. First, both the input and the output operators must satisfy the usual commutation relations:

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \quad \left[\hat{b}, \hat{b}^\dagger\right] = 1 \tag{5.3}$$

Second, the linearity of the amplifier and the fact that it is phase preserving (i.e. both signal quadratures are amplified the same way) implies a simple relation between the output operator $\hat{b}$ and the input operator $\hat{a}$:

$$\hat{b} = \sqrt{G}\hat{a}, \quad \hat{b}^\dagger = \sqrt{G}\hat{a}^\dagger, \tag{5.4}$$

where G is the dimensionless *"photon number gain"* of the amplifier. It is immediately clear however this expression cannot possibly be correct as written because it violates the fundamental bosonic commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$. We are therefore forced to write:

$$\hat{b} = \sqrt{G}\hat{a} + \hat{\mathcal{F}}, \quad \hat{b}^\dagger = \sqrt{G}\hat{a}^\dagger + \hat{\mathcal{F}}^\dagger, \tag{5.5}$$

where $\hat{\mathcal{F}}$ is an operator representing additional noise added by the amplifier. Based on the discussion of the previous subsection, we can anticipate what $\hat{\mathcal{F}}$ represents: it is noise associated with the additional degrees of freedom which must invariably be present in a phase-preserving amplifier.

As $\hat{\mathcal{F}}$ represents noise, it has a vanishing expectation value; in addition, one also assumes that this noise is uncorrelated with the input signal, implying $[\hat{\mathcal{F}}, \hat{a}] = [\hat{\mathcal{F}}, \hat{a}^\dagger] = 0$ and $\langle \hat{\mathcal{F}} \hat{a} \rangle = \langle \hat{\mathcal{F}} \hat{a}^\dagger \rangle = 0$. Insisting that $[\hat{b}, \hat{b}^\dagger] = 1$ thus yields

$$\left[\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\right] = 1 - G. \tag{5.6}$$

The question now becomes how small can we make the noise described by $\hat{\mathcal{F}}$? Using Eqs. (5.5), the noise at the amplifier output Δb is given by

$$
\begin{aligned}
(\Delta b)^2 &= G\,(\Delta a)^2 + \frac{1}{2}\left\langle \{\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\} \right\rangle \\
&\geq G\,(\Delta a)^2 + \frac{1}{2}\left| \left\langle [\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger] \right\rangle \right| \\
&\geq G\,(\Delta a)^2 + \frac{|G-1|}{2}.
\end{aligned} \tag{5.7}
$$

We have used here a standard inequality to bound the expectation of $\{\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\}$. The first term here is simply the amplified noise of the input, while the second term represents the noise added by the amplifier. Note that if there is no amplification (i.e. $G = 1$), there need not be any added noise. However, in the more relevant case of large amplification ($G \gg 1$), the added noise cannot vanish. It is useful to express the noise at the output as an equivalent noise at ("referred to") the input by simply dividing out the photon gain G. Taking the large-G limit, we have

$$
\frac{(\Delta b)^2}{G} \geq (\Delta a)^2 + \frac{1}{2}. \tag{5.8}
$$

Thus, we have a very simple demonstration that *an amplifier with a large photon gain must add at least half a quantum of noise to the input signal.* Equivalently, the minimum value of the added noise is simply equal to the zero-point noise associated with the input mode; the total output noise (referred to the input) is at least twice the zero point input noise. Note the complete analogy to the quantum limit we found for the added noise of an op-amp amplifier (c.f. Eqs. (4.11) and (4.33)).

As already discussed, the added noise operator $\hat{\mathcal{F}}$ is associated with additional degrees of freedom (beyond input and output modes) necessary for phase-preserving amplification. To see this more concretely, note that every linear amplifier is inevitably a *non-linear* system consisting of an energy source and a 'spigot' controlled by the input signal which redirects the energy source partly to the output channel and partly to some other channel(s). Hence there are inevitably other degrees of freedom involved in the amplification process beyond the input and output channels. An explicit example is the quantum parametric amplifier (see, e.g., Sec. V.C in (Clerk *et al.*, 2010)).

To see explicitly the role of the additional degrees of freedom, note first that for $G > 1$ the RHS of Eq. (5.6) is negative. Hence the simplest possible form for the added noise is:

$$
\hat{\mathcal{F}} = \sqrt{G-1}\,\hat{d}^\dagger, \quad \hat{\mathcal{F}}^\dagger = \sqrt{G-1}\,\hat{d}, \tag{5.9}
$$

where $\hat{d}$ and $\hat{d}^\dagger$ represent a single additional mode of the system. This is the minimum number of additional degrees of freedom that must inevitably be involved in the amplification process. Note that for this case, the inequality in Eq. (5.7) is satisfied as an equality, and the added noise takes on its minimum possible value. If instead we had, say, two additional modes (coupled inequivalently):

$$
\hat{\mathcal{F}} = \sqrt{G-1}(\cosh\theta\,\hat{d}_1^\dagger + \sinh\theta\,\hat{d}_2) \tag{5.10}
$$

it is straightforward to show that the added noise is inevitably larger than the minimum. This again can be interpreted in terms of wasted information, as the extra

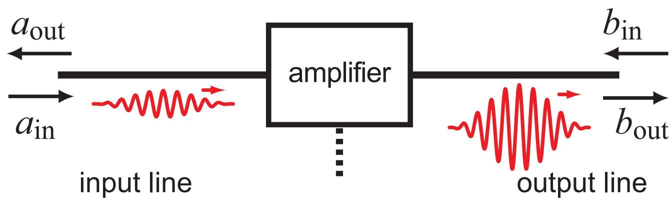

Fig. 5.1 (Color online) Schematic of a two-port bosonic amplifier. Both the input and outputs of the amplifier are attached to transmission lines. The incoming and outgoing wave amplitudes in the input (output) transmission line are labelled $\hat{a}_{\rm in}, \hat{a}_{\rm out}$ ($\hat{b}_{\rm in}, \hat{b}_{\rm out}$) respectively. The voltages at the end of the two lines ($\hat{V}_a, \hat{V}_b$) are linear combinations of incoming and outgoing wave amplitudes.

degrees of freedom are not being monitored as part of the measurement process and so information is being lost.

5.2 Bosonic Scattering Description of a Two-Port Amplifier

To highlight the differences between the op-amp and scattering modes of amplifier operation (and the quantum limit in each case), we will now consider a generic device that can be used in both modes. We will see explicitly that if we construct this device to be ideal as scattering amplifier, then it cannot be used as a quantum limited op-amp style amplifier. The basic reason is simple. Being ideal in the scattering mode means having an amplifier which is ideally suited for use with a signal source which is impedance matched to the amplifier input. In contrast, being ideal in the op-amp mode requires a signal impedance which is much smaller than the source impedance.

5.2.1 Scattering versus op-amp representations

In the bosonic scattering approach, a generic linear amplifier is modelled as a set of coupled bosonic modes. To make matters concrete, we will consider the specific case of a voltage amplifier with distinct input and output ports, where each port is a semi-infinite transmission line (see Fig. 5.1). We start by recalling that a quantum transmission line can be described as a set of non-interacting bosonic modes (see Appendix D of (Clerk *et al.*, 2010) for a quick review). Denoting the input transmission line with an a and the output transmission line with a b, the current and voltage operators in these lines may be written:

$$\hat{V}_q(t) = \int_0^\infty \frac{d\omega}{2\pi} \left(\hat{V}_q[\omega] e^{-i\omega t} + h.c. \right), \tag{5.11a}$$

$$\hat{I}_q(t) = \sigma_q \int_0^\infty \frac{d\omega}{2\pi} \left(\hat{I}_q[\omega] e^{-i\omega t} + h.c. \right), \tag{5.11b}$$

with

$$\hat{V}_q[\omega] = \sqrt{\frac{\hbar\omega}{2} Z_q}\left(\hat{q}_{\rm in}[\omega] + \hat{q}_{\rm out}[\omega]\right), \tag{5.12a}$$

$$\hat{I}_q[\omega] = \sqrt{\frac{\hbar\omega}{2Z_q}}\left(\hat{q}_{\rm in}[\omega] - \hat{q}_{\rm out}[\omega]\right). \tag{5.12b}$$

Here, q can be equal to a or b, and we have $\sigma_a = 1, \sigma_b = -1$. The operators $\hat{a}_{\rm in}[\omega], \hat{a}_{\rm out}[\omega]$ are bosonic annihilation operators; $\hat{a}_{\rm in}[\omega]$ describes an incoming wave in the input transmission line (i.e. incident on the amplifier) having frequency ω, while $\hat{a}_{\rm out}[\omega]$ describes an outgoing wave with frequency ω. The operators $\hat{b}_{\rm in}[\omega]$ and $\hat{b}_{\rm out}[\omega]$ describe analogous waves in the output transmission line. We can think of $\hat{V}_a$ as the input voltage to our amplifier, and $\hat{V}_b$ as the output voltage. Similarly, $\hat{I}_a$ is the current drawn by the amplifier at the input, and $\hat{I}_b$ the current drawn at the output of the amplifier. Finally, Z_a (Z_b) is the characteristic impedance of the input (output) transmission line.

As we have seen, amplification invariably requires additional degrees of freedom. Thus, to amplify a signal at a particular frequency ω, there will be $2N$ bosonic modes involved, where the integer N is necessarily larger than 2. Four of these modes are simply the frequency-ω modes in the input and output lines (i.e. $\hat{a}_{\rm in}[\omega]$,$\hat{a}_{\rm out}[\omega]$,$\hat{b}_{\rm in}[\omega]$ and $\hat{b}_{\rm out}[\omega]$). The remaining $2(N-2)$ modes describe auxiliary degrees of freedom involved in the amplification process; these additional modes could correspond to frequencies different from the signal frequency ω. The auxiliary modes can also be divided into incoming and outgoing modes. It is thus convenient to represent them as additional transmission lines attached to the amplifier; these additional lines could be semi-infinite, or could be terminated by active elements.

Scattering representation. In general, our generic two-port bosonic amplifier will be described by a $N \times N$ scattering matrix which determines the relation between the outgoing mode operators and incoming mode operators. The form of this matrix is constrained by the requirement that the output modes obey the usual canonical bosonic commutation relations. It is convenient to express the scattering matrix in a form which *only* involves the input and output lines explicitly:

$$\begin{pmatrix} \hat{a}_{\rm out}[\omega] \\ \hat{b}_{\rm out}[\omega] \end{pmatrix} = \begin{pmatrix} s_{11}[\omega] & s_{12}[\omega] \\ s_{21}[\omega] & s_{22}[\omega] \end{pmatrix} \begin{pmatrix} \hat{a}_{\rm in}[\omega] \\ \hat{b}_{\rm in}[\omega] \end{pmatrix} + \begin{pmatrix} \hat{\mathcal{F}}_a[\omega] \\ \hat{\mathcal{F}}_b[\omega] \end{pmatrix}. \tag{5.13}$$

Here $\hat{\mathcal{F}}_a[\omega]$ and $\hat{\mathcal{F}}_b[\omega]$ are each an unspecified linear combination of the incoming auxiliary modes introduced above. They thus describe noise in the outgoing modes of the input and output transmission lines which arises from the auxiliary modes involved in the amplification process. Similar to Eq. (5.5) in our discussion of a one port amplifier, the $\hat{\mathcal{F}}$ operators also ensure that canonical commutation relations are preserved.

In the quantum optics literature, one typically views Eq. (5.13) as the defining equation of the amplifier; we will call this the scattering representation of our amplifier. The representation is best suited to understanding the amplifier when used in the scattering mode of operation. In this mode, the experimentalist ensures that $\langle \hat{a}_{\rm in}[\omega] \rangle$

is precisely equal to the signal to be amplified, *irrespective* of what is coming out of the amplifier. Similarly, the output signal from the amplifier is the amplitude of the outgoing wave in the output line, $\langle \hat{b}_{\mathrm{out}}[\omega] \rangle$. If we focus on $\hat{b}_{\mathrm{out}}$, we have precisely the same situation as described in Sec. 5.8, where we presented the Haus-Caves derivation of the quantum limit (cf. Eq. (5.5)). It thus follows that in the scattering mode of operation, the matrix element $s_{21}[\omega]$ represents the gain of our amplifier at frequency ω, $|s_{21}[\omega]|^2$ the corresponding "photon number gain", and $\hat{\mathcal{F}}_b$ the added noise operator of the amplifier. The operator $\hat{\mathcal{F}}_a$ represents the back-action noise in the scattering mode of operation; this back-action has no effect on the added noise of the amplifier in the scattering mode.

Op-amp representation. In the op-amp amplifier mode of operation, the input and output signals are not simply incoming/outgoing wave amplitudes; thus, the scattering representation is not an optimal description of our amplifier. The system we are describing here is a voltage amplifier: thus, in the op-amp mode, the experimentalist would ensure that the voltage at the end of the input line ($\hat{V}_a$) is equal to the signal to be amplified, and would read out the voltage at the end of the output transmission line ($\hat{V}_b$) as the output of the amplifier. From Eq. (5.11a), we see that this implies that the amplitude of the wave going into the amplifier, a_{in}, will depend on the amplitude of the wave exiting the amplifier, a_{out}.

Thus, if we want to use our amplifier as a voltage amplifier, we would like to find a description which is more tailored to our needs than the scattering representation of Eq. (5.13). This can be found by simply re-expressing the scattering matrix relation of Eq. (5.13) in terms of voltages and currents. The result will be what we term the "op amp" representation of our amplifier, a representation which is standard in the discussion of classical amplifiers (see, e.g., (Boylestad and Nashelsky, 2006)). In this representation, one views $\hat{V}_a$ and $\hat{I}_b$ as inputs to the amplifier: $\hat{V}_a$ is set by whatever we connect to the amplifier input, while $\hat{I}_b$ is set by whatever we connect to the amplifier output. In contrast, the outputs of our amplifier are the voltage in the output line, $\hat{V}_b$, and the current drawn by the amplifier at the input, $\hat{I}_a$.

Using Eqs. (5.11a) and (5.11b), and suppressing frequency labels for clarity, Eq. (5.13) may be written explicitly in terms of the voltages and current in the input $(\hat{V}_a, \hat{I}_a)$ and output $(\hat{V}_b, \hat{I}_b)$ transmission lines:

$$\begin{pmatrix} \hat{V}_b \\ \hat{I}_a \end{pmatrix} = \begin{pmatrix} \lambda_V & -Z_{\mathrm{out}} \\ \frac{1}{Z_{\mathrm{in}}} & \lambda'_I \end{pmatrix} \begin{pmatrix} \hat{V}_a \\ \hat{I}_b \end{pmatrix} + \begin{pmatrix} \lambda_V \cdot \hat{\tilde{V}} \\ \hat{\tilde{I}} \end{pmatrix}. \tag{5.14}$$

The coefficients in the above matrix are completely determined by the scattering matrix of Eq. (5.13) (see Eqs. (5.17) below); moreover, they are familiar from the discussion of a voltage amplifier in Sec. 4.7. $\lambda_V[\omega]$ is the voltage gain of the amplifier, $\lambda'_I[\omega]$ is the reverse current gain of the amplifier, Z_{out} is the output impedance, and Z_{in} is the input impedance. The last term on the RHS of Eq. (5.14) describes the two familiar kinds of amplifier noise. $\hat{\tilde{V}}$ is the usual voltage noise of the amplifier (referred back to the amplifier input), while $\hat{\tilde{I}}$ is the usual current noise of the amplifier. Recall that in this standard description of a voltage amplifier (cf. Sec. 4.7), $\tilde{I}$ represents the back-action of the amplifier: the system producing the input signal responds to these current fluctuations, resulting in an additional fluctuation in the input signal going into the amplifier. Similarly, $\lambda_V \cdot \tilde{V}$ represents the intrinsic output noise of the amplifier: this contribution to the total output noise does not depend on properties of the input signal. Note that we are using a sign convention where a positive $\langle \hat{I}_a \rangle$ indicates a current flowing *into* the amplifier at its input, while a positive $\langle \hat{I}_b \rangle$ indicates a current flowing *out of* the amplifier at its output. Also note that the operators $\hat{V}_a$ and $\hat{I}_b$ on the RHS of Eq. (5.14) will have noise; this noise is entirely due to the systems attached to the input and output of the amplifier, and as such, should not be included in what we call the added noise of the amplifier.

Additional important properties of our amplifier follow immediately from quantities in the op-amp representation. As discussed in Sec. 4.3, the most important measure of gain in our amplifier is the dimensionless power gain. This is the ratio between power dissipated at the output to that dissipated at the input, taking the output current I_B to be V_B/Z_{out}:

$$G_P \equiv \frac{(\lambda_V)^2}{4} \frac{Z_{\mathrm{in}}}{Z_{\mathrm{out}}} \cdot \left(1 + \frac{\lambda_V \lambda_I'}{2} \frac{Z_{\mathrm{in}}}{Z_{\mathrm{out}}}\right)^{-1} \tag{5.15}$$

Another important quantity is the loaded input impedance: what is the input impedance of the amplifier in the presence of a load attached to the output? In the presence of reverse current gain $\lambda_I' \neq 0$, the input impedance will depend on the output load. Taking the load impedance to be Z_{load}, some simple algebra yields:

$$\frac{1}{Z_{\mathrm{in,loaded}}} = \frac{1}{Z_{\mathrm{in}}} + \frac{\lambda_I' \lambda_V}{Z_{\mathrm{load}} + Z_{\mathrm{out}}} \tag{5.16}$$

It is of course undesirable to have an input impedance which depends on the load. Thus, we see yet again that it is undesirable to have appreciable reverse gain in our amplifier.

Converting between representations. Some straightforward algebra now lets us express the op-amp parameters appearing in Eq. (5.14) in terms of the scattering matrix appearing in Eq. (5.13):

$$\lambda_V = 2\sqrt{\frac{Z_b}{Z_a}} \frac{s_{21}}{D}, \tag{5.17a}$$

$$\lambda_I' = 2\sqrt{\frac{Z_b}{Z_a}} \frac{s_{12}}{D}, \tag{5.17b}$$

$$Z_{\mathrm{out}} = Z_b \frac{(1 + s_{11})(1 + s_{22}) - s_{12}s_{21}}{D}, \tag{5.17c}$$

$$\frac{1}{Z_{\mathrm{in}}} = \frac{1}{Z_a} \frac{(1 - s_{11})(1 - s_{22}) - s_{12}s_{21}}{D}. \tag{5.17d}$$

All quantities are evaluated at the same frequency ω, and D is defined as

$$D = (1 + s_{11})(1 - s_{22}) + s_{12}s_{21}. \tag{5.18}$$

Further, the voltage and current noises in the op-amp representation are simple linear combinations of the noises $\hat{\mathcal{F}}_a$ and $\hat{\mathcal{F}}_b$ appearing in the scattering representation:

$$\begin{pmatrix} \hat{\tilde{V}} \\ Z_a \cdot \hat{\tilde{I}} \end{pmatrix} = \sqrt{2\hbar\omega Z_a} \begin{pmatrix} -\frac{1}{2} & \frac{1+s_{11}}{2s_{21}} \\ \frac{s_{22}-1}{D} & -\frac{s_{12}}{D} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{F}}_a \\ \hat{\mathcal{F}}_b \end{pmatrix}. \tag{5.19}$$

Again, all quantities above are evaluated at frequency ω.

Eq. (5.19) immediately leads to an important conclusion and caveat: *what one calls the "back-action" and "added noise" in the scattering representation (i.e. $\mathcal{F}_a$ and $\mathcal{F}_b$) are not the same as the "back-action" and "added noise" defined in the usual op-amp representation.* For example, the op-amp back-action $\hat{\tilde{I}}$ does not in general coincide with the $\hat{\mathcal{F}}_a$, the back-action in the scattering picture. If we are indeed interested in using our amplifier as a voltage amplifier, we are interested in the total added noise of our amplifier *as defined in the op-amp representation*. As we saw in Sec. 4.7 (cf. Eq. (4.34)), this quantity involves both the noises $\hat{\tilde{I}}$ and $\hat{\tilde{V}}$.

5.2.2 A seemingly ideal two-port amplifier

Scattering versus op-amp quantum limit. In this subsection we demonstrate that an amplifier which is "ideal" and minimally complex when used in the scattering operation mode fails, when used as a voltage op-amp, to have a quantum limited noise temperature. The system we look at is very similar to the amplifier considered by (Grassia, 1998), though our conclusions are somewhat different than those found there.

In the scattering representation, one might guess that an "ideal" amplifier would be one where there are no reflections of signals at the input and output, and no way for incident signals at the output port to reach the input. In this case, Eq. (5.13) takes the form

$$\begin{pmatrix} \hat{a}_{\text{out}} \\ \hat{b}_{\text{out}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sqrt{G} & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{\text{in}} \\ \hat{b}_{\text{in}} \end{pmatrix} + \begin{pmatrix} \hat{\mathcal{F}}_a \\ \hat{\mathcal{F}}_b \end{pmatrix}, \tag{5.20}$$

where we have defined $\sqrt{G} \equiv s_{21}$. All quantities above should be evaluated at the same frequency ω; for clarity, we will omit writing the explicit ω dependence of quantities throughout this section.

Turning to the op-amp representation, the above equation implies that our amplifier has no reverse gain, and that the input and output impedances are simply given by the impedances of the input and output transmission lines. From Eqs. (5.17), we have:

$$\lambda_V = 2\sqrt{\frac{Z_b}{Z_a}G}, \tag{5.21a}$$

$$\lambda'_I = 0, \tag{5.21b}$$

$$Z_{\text{out}} = Z_b, \tag{5.21c}$$

$$\frac{1}{Z_{\text{in}}} = \frac{1}{Z_a}. \tag{5.21d}$$

We immediately see that our amplifier looks less ideal as an op-amp. The input and output impedances are the same as those of the input and output transmission line. However, for an ideal op-amp, we would have liked $Z_{\rm in} \to \infty$ and $Z_{\rm out} \to 0$.

Also of interest are the expressions for the amplifier noises in the op-amp representation:

$$\begin{pmatrix} \hat{\tilde{V}} \\ Z_a \cdot \hat{\tilde{I}} \end{pmatrix} = -\sqrt{2\hbar\omega Z_a} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{G}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{F}}_a \\ \hat{\mathcal{F}}_b \end{pmatrix}. \tag{5.22}$$

As $s_{12} = 0$, the back-action noise is the same in both the op-amp and scattering representations: it is determined completely by the noise operator $\hat{\mathcal{F}}_a$. However, the voltage noise (i.e. the intrinsic output noise) involves *both* $\hat{\mathcal{F}}_a$ and $\hat{\mathcal{F}}_b$. We thus have the unavoidable consequence that there will be correlations in $\hat{\tilde{I}}$ and $\hat{\tilde{V}}$.

To make further progress, we note again that commutators of the noise operators $\hat{\mathcal{F}}_a$ and $\hat{\mathcal{F}}_b$ are completely determined by Eq. (5.20) and the requirement that the output operators obey canonical commutation relations. We thus have:

$$\left[\hat{\mathcal{F}}_a, \hat{\mathcal{F}}_a^\dagger\right] = 1, \tag{5.23a}$$

$$\left[\hat{\mathcal{F}}_b, \hat{\mathcal{F}}_b^\dagger\right] = 1 - |G|, \tag{5.23b}$$

$$\left[\hat{\mathcal{F}}_a, \hat{\mathcal{F}}_b\right] = \left[\hat{\mathcal{F}}_a, \hat{\mathcal{F}}_b^\dagger\right] = 0. \tag{5.23c}$$

We will be interested in the limit of a large power gain, which requires $|G| \gg 1$. A minimal solution to the above equations would be to have the noise operators determined by two independent (i.e. mutually commuting) auxiliary input mode operators u_{in} and $v_{in}^\dagger$:

$$\hat{\mathcal{F}}_a = \hat{u}_{\rm in}, \tag{5.24}$$

$$\hat{\mathcal{F}}_b = \sqrt{|G|-1}\,\hat{v}_{\rm in}^\dagger. \tag{5.25}$$

Further, to minimize the noise of the amplifier, we take the operating state of the amplifier to be the vacuum for both these modes. With these choices, our amplifier is in exactly the minimal form described by (Grassia, 1998): an input and output line coupled to a negative resistance box and an auxiliary "cold load" via a four-port circulator (see Fig. 5.2). The negative resistance box is nothing but the single-mode bosonic amplifier discussed in Sec. 5.1; an explicit realization of this element would be a non-degenerate parametric amplifier (see, e.g., Sec. V.C in (Clerk *et al.*, 2010)). The "cold load" is a semi-infinite transmission line which models dissipation due to a resistor at zero-temperature.

Note that within the scattering picture, one would conclude that our amplifier is ideal: in the large gain limit, the noise added by the amplifier to $\hat{b}_{\rm out}$ corresponds to a single quantum at the input:

$$\frac{\left\langle\left\{\hat{\mathcal{F}}_b, \hat{\mathcal{F}}_b^\dagger\right\}\right\rangle}{|G|} = \frac{|G|-1}{|G|}\left\langle\left\{\hat{v}_{\rm in}^\dagger, \hat{v}_{\rm in}\right\}\right\rangle \to 1 \tag{5.26}$$

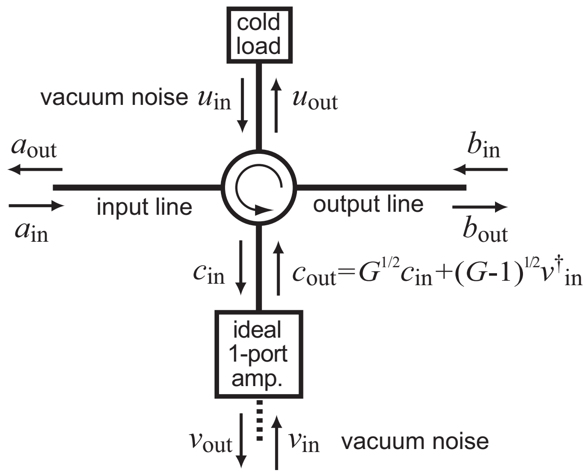

Fig. 5.2 Schematic of a "minimal" two-port amplifier which reaches the quantum limit in the scattering mode of operation, but misses the quantum limit when used as a weakly-coupled op-amp. See text for further description

This however is *not* the quantity which interests us: as we want to use this system as a voltage op-amp, we would like to know if the noise temperature *defined in the op-amp picture* is as small as possible. We are also usually interested in the case of a signal which is weakly coupled to our amplifier; here, weak-coupling means that the input impedance of the amplifier is much larger than the impedance of the signal source (i.e. $Z_{\text{in}} \gg Z_{\text{s}}$). In this limit, the amplifier only slightly increases the total damping of the signal source.

To address whether we can reach the op-amp quantum limit in the weak-coupling regime, we can make use of the results of the general theory presented in Sec. 4.7. In particular, we need to check whether the quantum noise constraint of Eq. (4.39) is optimized, as this is a prerequisite for reaching the (weak-coupling) quantum limit. Thus, we need to calculate the *symmetrized* spectral densities of the current and voltage noises, and their cross-correlation. It is easy to confirm from the definitions of Eq. (5.11a) and (5.11b) that these quantities take the form:

$$\bar{S}_{VV}[\omega] = \frac{\left\langle \left\{ \hat{\tilde{V}}[\omega], \hat{\tilde{V}}^{\dagger}[\omega'] \right\} \right\rangle}{4\pi\delta(\omega - \omega')}, \tag{5.27a}$$

$$\bar{S}_{II}[\omega] = \frac{\left\langle \left\{ \hat{\tilde{I}}[\omega], \hat{\tilde{I}}^{\dagger}[\omega'] \right\} \right\rangle}{4\pi\delta(\omega - \omega')}, \tag{5.27b}$$

$$\bar{S}_{VI}[\omega] = \frac{\left\langle \left\{ \hat{\tilde{V}}[\omega], \hat{\tilde{I}}^{\dagger}[\omega'] \right\} \right\rangle}{4\pi\delta(\omega - \omega')}. \tag{5.27c}$$

The expectation values here are over the operating state of the amplifier; we have chosen this state to be the vacuum for the auxiliary mode operators $\hat{u}_{\text{in}}$ and $\hat{v}_{\text{in}}$ to minimize the noise.

Taking $|s_{21}| \gg 1$, and using Eqs. (5.24) and (5.25), we have:

$$\bar{S}_{VV}[\omega] = \frac{\hbar\omega Z_a}{4}\left(\sigma_{uu} + \sigma_{vv}\right) = \frac{\hbar\omega Z_a}{2}, \tag{5.28a}$$

$$\bar{S}_{II}[\omega] = \frac{\hbar\omega}{Z_a}\sigma_{uu} = \frac{\hbar\omega}{Z_a}, \tag{5.28b}$$

$$\bar{S}_{VI}[\omega] = \frac{\hbar\omega}{2}\sigma_{uu} = \frac{\hbar\omega}{2}, \tag{5.28c}$$

where we have defined

$$\sigma_{ab} \equiv \left\langle \hat{a}\hat{b}^\dagger + \hat{b}^\dagger\hat{a} \right\rangle, \tag{5.29}$$

and have used the fact that there cannot be any correlations between the operators u and v in the vacuum state (i.e. $\langle \hat{u}\hat{v}^\dagger \rangle = 0$).

It follows immediately from the above equations that our minimal amplifier does not optimize the quantum noise constraint of Eq. (4.39):

$$\bar{S}_{VV}[\omega]\bar{S}_{II}[\omega] - \left[\mathrm{Im}\ \bar{S}_{VI}\right]^2 = 2 \times \left(\frac{\hbar\omega}{2}\right)^2. \tag{5.30}$$

The noise product $\bar{S}_{VV}\bar{S}_{II}$ is precisely twice the quantum-limited value. As a result, the general theory of Sec. 4.7 tells us *if one couples an input signal weakly to this amplifier (i.e. $Z_\mathrm{s} \ll Z_\mathrm{in}$), it is impossible to reach the quantum limit on the added noise.* Thus, while our amplifier is ideal in the scattering mode of operation (cf. Eq. (5.26)), it fails to reach the quantum limit when used in the weak-coupling, op-amp mode of operation. Our amplifier's failure to have "ideal" quantum noise also means that if we tried to use it to do QND qubit detection, the resulting back-action dephasing would be twice as large as the minimum required by quantum mechanics.

5.2.3 Why is the op-amp quantum limit is not achieved?

Note that if one substitutes the expressions in Eqs. (5.28) (including the real cross-correlation noise) in the classical lower bound for T_N given in Eq. (4.40), one finds the disturbing conclusion that $T_N[\omega]$ not only reaches the quantum limit, but surpasses it: one finds $T_N[\omega] = (\sqrt{2} - 1)\,(\hbar\omega/2)$. Before one gets too excited (or alarmed) by this, note that this expression was derived for a noise-matched source impedance ($Z_\mathrm{s} = Z_\mathrm{N}$), as well as assuming $Z_\mathrm{s} \ll Z_\mathrm{in}$. The noise impedance here however is given by

$$Z_\mathrm{N} = \sqrt{\frac{\bar{S}_{VV}}{\bar{S}_{II}}} = \frac{Z_\mathrm{in}}{\sqrt{2}} \tag{5.31}$$

Thus, the weak coupling condition of $Z_\mathrm{s} \ll Z_\mathrm{in}$ is not fulfilled, and the Eq. (4.40) for T_N is not valid.

One can instead directly calculate the total added noise of the amplifier (as always, referred back to the input), for a finite ratio of Z_s/Z_{in}. This added noise takes the form:

$$\hat{\tilde{V}}_{tot} = -\left(\frac{Z_s Z_a}{Z_s + Z_a}\right)\hat{\tilde{I}} + \hat{\tilde{V}} \tag{5.32}$$

Note that this classical-looking equation can be rigorously justified within the full quantum theory if one starts with a full description of the amplifier and the signal source (e.g. a parallel LC oscillator attached in parallel to the amplifier input). Plugging in the expressions for $\hat{\tilde{I}}$ and $\hat{\tilde{V}}$, we find:

$$\begin{aligned}
\hat{\tilde{V}}_{tot} &= \sqrt{\frac{\hbar\omega}{2}}\left[\left(\frac{Z_s Z_a}{Z_s + Z_a}\right)\left(\frac{2}{\sqrt{Z_a}}\hat{u}_{in}\right) - \sqrt{Z_a}(\hat{u}_{in} - \hat{v}^\dagger_{in})\right] \\
&= \sqrt{\frac{\hbar\omega Z_a}{2}}\left[\left(\frac{Z_s - Z_a}{Z_s + Z_a}\right)\hat{u}_{in} - \hat{v}^\dagger_{in}\right]
\end{aligned} \tag{5.33}$$

Thus, if one impedance matches the source (i.e. tunes Z_s to $Z_a = Z_{in}$), the mode $\hat{u}_{in}$ does not contribute to the total added noise, and one reaches the appropriately-defined quantum limit on the added noise[2]. One is of course very far from the weak coupling condition needed for the op-amp mode, and is thus effectively operating in the scattering mode.

Returning to the more interesting case of a weak amplifier-signal coupling, for $Z_s \ll Z_a \sim Z_{in}$, one finds that the noise temperature is a large factor Z_{in}/Z_s bigger than the quantum limited value. It is possible to understand the failure to reach the quantum limit in this weak-coupling limit heuristically. To that end, note again that the amplifier noise cross-correlation $\bar{S}_{IV}$ does not vanish in the large-gain limit (cf. Eq. (5.28c)). Correlations between the two amplifier noises represent a kind of information, as by making use of them, we can improve the performance of the amplifier. It is easy to take advantage of out-of-phase correlations between $\tilde{I}$ and $\tilde{V}$ (i.e. Im $\bar{S}_{VI}$) by simply tuning the phase of the source impedance (cf. Eq. (4.38)). However, one cannot take advantage of in-phase noise correlations (i.e. Re $\bar{S}_{VI}$) as easily. To take advantage of the information here, one needs to modify the amplifier itself. By feeding back some of the output voltage to the input, one could effectively cancel out some of the back-action current noise $\tilde{I}$ and thus reduce the overall magnitude of $\bar{S}_{II}$. Hence, the unused information in the cross-correlator Re $\bar{S}_{VI}$ represents a kind of wasted information: had we made use of these correlations via a feedback loop, we could have reduced the noise temperature and increased the information provided by our amplifier. (Clerk *et al.*, 2010) discusses this point further, and shows how feedback may be implemented in the above amplifier by introducing reflections in the transmission lines leading to the circulator.

[2]One must be careful here in defining the added noise, as when Z_s is not much smaller than Z_{in}, the amplifier will appreciably increase the dissipation of the signal source and reduce its intrinsic thermal fluctuations. This reduction should *not* be included in the definition of the added noise. This is analogous to linear position detection in the case where the backaction damping is large; the cooling effect of the backaction damping is not included in the definition of the quantum limit, see. Eq. (4.12)

References

Averin, D. (2003). Linear quantum measurements. In *Quantum Noise in Mesoscopic Systems* (ed. Y. Nazarov), pp. 205–228. Kluwer, Amsterdam.

Averin, D. V. (2000). Quantum computing and quantum measurement with mesoscopic josephson junctions. *Fortschr. Phys.*, **48**, 1055.

Blanter, Ya. M. and Büttiker, M. (2000). Shot noise in mesoscopic conductors. *Phys. Rep.*, **336**, 1.

Blencowe, Miles P, Imbers, J, and Armour, A D (2005). Dynamics of a nanomechanical resonator coupled to a superconducting single-electron transistor. *New J. Phys.*, **7**, 236.

Boylestad, Robert and Nashelsky, Louis (2006). *Electronic Devices and Circuit Theory* (9 edn). Prentice Hall, New Jersey.

Braginsky, V. B. and Khalili, F. Y. (1996). Quantum nondemolition measurements: the route from toys to tools. *Rev. Mod. Phys.*, **68**, 1.

Bruus, Henrik and Flensberg, Karsten (2004). *Many-body quantum theory in condensed matter physics: a introduction.* Oxford University Press, Oxford.

Caves, Carlton M. (1982). Quantum limits on noise in linear amplifiers. *Phys. Rev. D*, **26**, 1817–1839.

Clerk, Aashish and Bennett, S (2005). Quantum nanoelectromechanics with electrons, quasi-particles and Cooper pairs: effective bath descriptions and strong feedback effects. *New J. Phys.*, **7**, 238.

Clerk, A. A. (2004). Quantum-limited position detection and amplification: A linear response perspective. *Phys. Rev. B*, **70**, 245306.

Clerk, A. A., Devoret, M. H., Girvin, S. M., Marquardt, F., and Schoelkopf, R. J. (2010). Introduction to quantum noise, measurement and amplification. *Rev. Mod. Phys.*, **82**, 1155.

Clerk, A. A., Girvin, S. M., and Stone, A. D. (2003). Quantum-limited measurement and information in mesoscopic detectors. *Phys. Rev. B*, **67**, 165324.

Devoret, M. H. and Schoelkopf, R. J. (2000). Amplifying quantum signals with the single-electron transistor. *Nature (London)*, **406**, 1039–1046.

Elzerman, JM, Hanson, R, van Beveren, LHW, Witkamp, B, Vandersypen, L. M. K, and Kouwenhoven, LP (2004, Jan). Single-shot read-out of an individual electron spin in a quantum dot. *Nature*, **430**(6998), 431–435.

Field, M, Smith, CG, Pepper, M, Ritchie, DA, Frost, JEF, Jones, GAC, and Hasko, DG (1993, Jan). Measurements of coulomb blockade with a noninvasive voltage probe. *Phys. Rev. Lett.*, **70**(9), 1311–1314.

Gottfried, K. (1966). *Quantum Mechanics, Volume I: Fundamentals.* W. A. Benjamin, Inc.

Grassia, F. (1998). *Fluctuations quantiques et thermiques dans les transducteurs* *electromecaniques*. Ph.D. thesis, Universite Pierre et Marie Curie.

Haus, H. A. (2000). *Electromagnetic Noise and Quantum Optical Measurements*. Springer, New York.

Haus, H. A. and Mullen, J. A. (1962). Quantum noise in linear amplifiers. *Phys. Rev.*, **128**, 2407.

Heffner, H. (1962). The fundamental noise limit of linear amplifiers. *Proc. IRE*, **50**, 1604.

Heisenberg, Werner (1930). *The Physical Principles of the Quantum Theory*. Dover, New York.

Korotkov, Alexander N. (1999). Continuous quantum measurement of a double dot. *Phys. Rev. B*, **60**, 5737–5742.

Korotkov, A. N. and Averin, D. V. (2001). Continuous weak measurement of quantum coherent oscillations. *Phys. Rev. B*, **64**, 165310.

Laflamme, C and Clerk, A (2011, March). Quantum-limited amplification with a nonlinear cavity detector. *Phys. Rev. A*, **83**(3), 033803.

Makhlin, Yuriy, Schon, Gerd, and Shnirman, Alexander (2001). Quantum-state engineering with josephson-junction devices. *Rev. Mod. Phys.*, **73**, 357.

Marquardt, F., Chen, J. P., Clerk, A. A., and Girvin, S. M. (2007). Quantum theory of cavity-assisted sideband cooling of mechanical motion. *Phys. Rev. Let.*, **99**, 093902.

Pathria, R. K. (1996). *Statistical Mechanics* (Second edn). Elsevier, Amsterdam.

Petta, JR, Johnson, AC, Marcus, Charles, Hanson, MP, and Gossard, AC (2004, Jan). Manipulation of a single charge in a double quantum dot. *Phys. Rev. Lett.*, **93**(18), 186802.

Pilgram, S. and Büttiker, M. (2002). Efficiency of mesoscopic detectors. *Phys. Rev. Lett.*, **89**, 200401.

Poggio, M., Jura, M. P., Degen, C. L., Topinka, M. A., Mamin, H. J., Goldhaber-Gordon, D., and Rugar, D. (2008). An off-board quantum point contact as a sensitive detector of cantilever motion. *Nature Phys.*, **4**, 635.

Sprinzak, D., Buks, E., Heiblum, M., and Shtrikman, H. (2000). Controlled dephasing of electrons via a phase sensitive detector. *Phys. Rev. Lett.*, **84**, 5820.

Wilson-Rae, I., Nooshi, N., Zwerger, W., and Kippenberg, T. J. (2007). Theory of ground state cooling of a mechanical oscillator using dynamical back-action. *Phys. Rev. Lett.*, **99**, 093901.

Young, C E and Clerk, A A (2010, May). Inelastic Backaction due to Quantum Point Contact Charge Fluctuations. *Phys. Rev. Lett.*, **104**(18), 186803.